Dynamical models for the optimal capacity level of an investment project

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DYNAMICAL MODELS FOR THE OPTIMAL CAPACITY LEVEL OF AN INVESTMENT PROJECT

by

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Abstract

This thesis is concerned with the solution of two stochastic control problems. In each case, we consider the problem of determining the optimal investment level that a firm should maintain in the presence of random price and/or demand fluctuations. We model market uncertainty by means of a geometric Brownian motion, and we consider general running payoff functions. The first model allows for capacity expansion as well as for capacity reduction, with each of these actions being associated with proportional costs. The resulting optimisation problem takes the form of a singular stochastic control problem that we solve explicitly. We illustrate our results by means of the so-called Cobb-Douglas production function. The second one permits capacity increases only, each associated with a fixed and a proportional cost. The resulting optimisation problem takes the form of a two-dimensional impulse control problem that we explicitly solve. The problems that we study present models, the associated Hamilton-Jacobi-Bellman equation of which admits a classical solution that conforms with the underlying economic intuition but does not necessarily identify with the corresponding value function, which may be identically equal to ∞. Thus, our models provide a situation that highlights the need for rigorous mathematical analysis when addressing stochastic optimisation applications in finance and economics, as well as in other fields.

To my mother and father

Contents

Introduction

In this thesis we solve two stochastic control problems resulting from the subject of determining the optimal capacity level of a given investment project producing a single commodity and operating within a random economic environment. In particular, we consider an investment project that yields payoff at a rate that, at any time t, is dependent both on its installed capacity level and an underlying economic indicator X_t which is modelled by a geometric Brownian motion and satisfies the SDE

$$dX_t = bX_t\,dt + \sqrt{2}\sigma X_t\,dW_t, \quad X_0 = x > 0, \tag{1}$$

for some constants b and σ. In practice, X_t can be the price of or the demand for the project's unique output commodity. The project's management control only the project's capacity level. The objective is to determine the project's capacity level that maximises an associated expected, discounted payoff flow.

A typical capacity expansion model such as the one solved by Kobila [K93] is driven by both a state process, which captures the randomness associated with the project and satisfies an SDE such as equation (1), and a capacity process which can be described as follows

$$Y_t = y + \xi_t^+, \quad Y_0 = y \geq 0,$$

where $y \geq 0$ is the project's initial capacity and ξ^+ models cumulative capacity increases. The project seeks a policy to maximise a performance criterion which, for each decision policy ξ^+, is given by

$$J_{x,y}(\xi^+) = \liminf_{T\to\infty} E\left[\int_0^T e^{-rt}h(X_t, Y_t)\,dt - K^+\int_{[0,T]} e^{-rt}\,d\xi_t^+\right],$$

where $h : \mathcal{S} \to \mathbb{R}$ is a given payoff function, $r > 0$ is the discount factor and K^+ is a constant which models the cost associated with increasing the project's capacity level.

The capacity expansion characterised above is of the irreversible type and has attracted significant interest in the literature. For example, Davis, Dempster, Sethi and Vermes [DDSV87] considered a model in which the decision maker controls the rate of investment in the current project (see also Davis [D93]) whereas Wang [W03] modelled the industry demand by a double exponential jump diffusion process. Other important contributions include Øksendal [Ø00], Chiarolla and Haussmann [CH05] and Bank [B05].

In Chapter 1, we consider a stochastic system the state of which is modelled by (1). The capacity process Y_t, namely its rate of output, can be increased or decreased at any time and at given proportional costs, and is given by

$$Y_t = y + \xi_t^+ - \xi_t^-, \quad Y_0 = y \geq 0, \tag{2}$$

where $y \geq 0$ is the project's initial capacity. Meanwhile, ξ^+, ξ^- are càglàd, increasing processes which model cumulative capacity increases, decreases respectively. The project's management is presented with the set of all decision strategies available, namely the set of all pairs (ξ^+, ξ^-) such that $Y_t \geq 0$, for all $t \geq 0$. With each decision policy, we associate the performance criterion

$$J_{x,y}(\xi^+, \xi^-) = E\left[\int_0^\infty e^{-rt} h(X_t, Y_t)\, dt - K^+ \int_{[0,\infty[} e^{-rt}\, d\xi_t^+ - K^- \int_{[0,\infty[} e^{-rt}\, d\xi_t^-\right],$$

where $h : \mathcal{S} \to \mathbb{R}$ is a given function, and K^+, K^- are constants. Here, h models the running payoff resulting from the project's operation, and K^+ (resp., K^-) models the costs associated with increasing (resp., decreasing) the project's capacity level. The objective is to maximise this performance index over all admissible decision strategies (ξ^+, ξ^-).

Abel and Eberly [AE96] considered a model involving both expansion and reduction of a project's capacity level. These authors assume that the rate at which the project yields payoff is modelled by a constant elasticity Cobb-Douglas production function. Our model considers much more general running payoff functions that include the Cobb-Douglas production functions as special cases, and allow for the more realistic situation where a running cost proportional to the project's installed capacity (reflecting, e.g., labour costs) is also included (see Example 1). Also, Guo and Pham [GP05] consider a related partially reversible investment model with entry decisions and a general running payoff function. The model that these authors consider is fundamentally different from the one considered by Abel and Eberly [AE96] or the one that we study here because, e.g., it is one-dimensional rather than two-dimensional.

Our analysis, which leads to results of an explicit analytic nature, involves the derivation of tight conditions for the project's value function to be finite. The fact that simple choices for the project's running payoff function lead to unique solutions to the associated free-boundary problem that conform with standard economic intuition but are associated with value functions that are identically equal to ∞ presents a most interesting feature of our analysis (see Remark 3; also, note that this pathological situation does not arise in the context of the special case studied by Abel and Eberly [AE96]). Indeed, this possibility stresses the fact that treating optimisation models related to investment decision making in a "formal" way, which is often the case in the economics literature, can lead to erroneous conclusions and can suggest the adoption of potentially disastrous policies.

The model solved in Chapter 1 has the structure of a singular stochastic control problem. Singular stochastic control, namely operating some control variable(s) continuously in time, with an expected discounted criterion was first considered by Bather and Chernoff [BC67] who examined a model of spaceship control. In their seminal paper, Beneš, Shepp and Witsenhausen [BSW80] were the first to solve rigorously an example of finite fuel singular control. Since then the area has attracted considerable interest in the literature. Karatzas [K93], Harrison and Taksar [HT83], Shreve, Lehoczky and Gavers [SLG84], Chow, Menaldi and Robin [CMR85], Sun [S87], Soner and Shreve [SS89], Ma [M92], Zhu [Z92], and Fleming and Soner [FS93, Chapter VIII] provide a list of further important contributions.

In Chapter 2 the state process X of the system also satisfies (1) but the project's capacity can only be increased at any time and we model capacity increases by jumps of an impulse control process Z. The capacity process Y is therefore given by

$$Y_t = y + Z_t, \quad Y_0 = y \geq 0, \tag{3}$$

where $y \geq 0$ is the project's initial capacity. For each decision policy Z, the objective is to maximise the following performance criterion

$$J_{x,y}(Z) = E\left[\int_0^\infty e^{-rt} h(X_t, Y_t)\, dt - \sum_{0 \leq t < \infty} e^{-rt}(K\Delta Z_t + c) 1_{\{\Delta Z_t > 0\}}\right]$$

where $h : \mathcal{S} \to \mathbb{R}$ is a given function and $K, c > 0$ are constants. Here, h models the running payoff resulting from the project's operation, and K, c provide a proportional and a fixed cost, respectively.

The irreversible capacity expansion models described above are associated with proportional costs only, and result in optimisation problems of singular control type. However, in Chapter 2, we assume that fixed costs are also incurred, each time that the project's capacity level is changed, resulting in a problem of the impulse control type. The concept of impulse control refers to an action exerted on some control variable(s) of a stochastic system, only at optimally chosen times of intervention, namely stopping times. This is particularly adequate in situations in which it is convenient not to perturb the system continuously, e.g FX rate. Jeanblanc-Picqué [JP93] investigated the problem of controlling in an impulsive way an FX rate, modelled by a Brownian motion with drift, so that it is confined within a fixed band $[a, b]$. Mundaca and Øksendal [MØ98] and Cadenillas and Zapatero [CZ99, CZ00] presented further contributions in this direction that also allow a central bank intervention policy that takes the form of absolutely continuous control of the drift of the underlying FX rate.

One of the first problems to allocate a fixed cost in addition to a cost which is proportional to the magnitude of the control applied was solved by Richard [R77]. In addition, he imposed carrying costs on the state of the system which was modelled by a homogeneous diffusion process. The principle of assuming both types of cost was applied in other optimal impulse control problems for various areas such as risky portfolios as in Eastham and Hastings [EH88] and cash management as in Baccarin [BN02].

Each of the two investment models is formulated rigorously. We derive the Hamilton-Jacobi-Bellman equations and we prove the appropriate verification theorems. The latter allow us to derive sufficient conditions, which conform with economic intuition for the associated optimisation problem to be well posed and to possess a finite value function, and we establish a number of estimates that we use in our analysis. In each case, we are presented with two free boundaries. In the singular control problem, the two boundaries separate the 'wait' and the 'investment' regions as well as the 'wait' and 'disinvestment' regions, respectively, so that the capacity process is either increased to the lower boundary or decreased to the upper boundary. In the impulse control problem, the capacity process is increased to the upper boundary whenever the initial capacity is in the 'investment' region or as soon as it hits the lower boundary. We impose assumptions on each of the models that ensure the uniqueness of the free boundary problem. We conjecture $C^{2,2}$ continuity at the free boundaries in the singular control model but in the impulsive one, we postulate $C^{1,1}$ and $C^{2,2}$ continuity at the lower and upper boundaries, respectively. This allows us to solve the free boundary problems and derive explicit solutions for the optimal capacity expansion strategies. We illustrate our results in Chapter 1 by means of the so called Cobb-Douglas function. In Chapter 2, such an illustration does not provide any additional insight into the structure of the problem.

The results should be of considerable interest not only to economists dealing with investment decisions, but also to others concerned with reversible and irreversible decisions taken in the face of future uncertainty, such as those concerning the management of natural resources, biological systems and the reduction of pollution.

Chapter 1

A model for reversible investment capacity expansion

1.1 Introduction

We consider the problem of determining in a dynamical way the optimal capacity level of a given investment project operating within a random economic environment. In particular, we consider an investment project that yields payoff at a rate that is dependent on its installed capacity level and on an underlying economic indicator such as the price of or the demand for the project's unique output commodity, which we model by a geometric Brownian motion. The project's capacity level can be increased or decreased at any time and at given proportional costs. The objective is to determine the project's capacity level that maximises the associated expected, discounted payoff flow. The resulting optimisation problem takes the form of a singular stochastic control problem that we solve explicitly. We illustrate our results by means of the so-called Cobb-Douglas production function.

The chapter is organised as follows. Section 1.2 is concerned with a rigorous formulation of the investment decision model that we study. In Section 1.3, we derive tight sufficient conditions, which conform with economic intuition, for the associated optimisation problem to possess a finite value function. We also establish a number of estimates that we use in our subsequent analysis. Section 1.4 is concerned with the proof of a verification theorem that provides sufficient conditions for the value function of our control problem to be identified with a solution to the associated dynamic programming or Hamilton-Jacobi-Bellman equation. Finally, we solve the optimisation problem considered in Section 1.5.

1.2 Problem formulation

We fix a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_t)$ satisfying the usual conditions of right continuity and augmentation by P-negligible sets, and carrying a standard, one-dimensional $(\mathcal{F}_t)$-Brownian motion W. We denote by $\mathcal{A}$ the family of all càglàd, $(\mathcal{F}_t)$-adapted, increasing processes ξ such that $\xi_0 = 0$.

We consider an investment project that produces a given commodity, and we assume that the project's capacity, namely its rate of output, can be controlled at any given time. We denote by Y_t the project's capacity at time t, and we model cumulative capacity increases (resp., decreases) by a process $\xi^+ \in \mathcal{A}$ (resp., $\xi^- \in \mathcal{A}$). In particular, given any times $0 \leq s \leq t$, $\xi^+_{t+} - \xi^+_s$ and $\xi^-_{t+} - \xi^-_s$ are the total capacity increase and decrease, respectively, incurred by the project management's decisions during the time interval $[s, t]$. The project's capacity process Y is therefore given by

$$Y_t = y + \xi^+_t - \xi^-_t, \quad Y_0 = y \geq 0, \tag{1.1}$$

where $y \geq 0$ is the project's initial capacity. Note that the assumptions that the processes $\xi^\pm$ are càglàd and $\xi^\pm_0 = 0$ imply that $Y_0 = y$. We make the assumption that the project's management controls only the project's capacity level. Accordingly, we denote by Π_y the set of all decision strategies available to the project's management, namely the set of all pairs (ξ^+, ξ^-) of processes $\xi^+, \xi^- \in \mathcal{A}$, such that $Y_t \geq 0$, for all $t \geq 0$.

We assume that all randomness associated with the project's operation can be captured by a state process X that satisfies the SDE

$$dX_t = bX_t\, dt + \sqrt{2}\sigma X_t\, dW_t, \quad X_0 = x > 0, \tag{1.2}$$

for some constants b and σ. In practice, X_t can be the price of one unit of the output commodity or an economic indicator reflecting, e.g., the output commodity's demand, at time t.

To simplify the notation, we define

$$\mathcal{S} = \left\{(x, y) \in \mathbb{R}^2 : \ x > 0, \ y \geq 0\right\},$$

so that $\mathcal{S}$ is the set of all possible initial conditions.

With each decision policy $(\xi^+, \xi^-) \in \Pi_y$ we associate the performance criterion

$$J_{x,y}(\xi^+, \xi^-) = E\left[\int_0^\infty e^{-rt} h(X_t, Y_t)\, dt - K^+ \int_{[0,\infty[} e^{-rt}\, d\xi_t^+ - K^- \int_{[0,\infty[} e^{-rt}\, d\xi_t^-\right], \tag{1.3}$$

where $h : \mathcal{S} \to \mathbb{R}$ is a given function, and

$$r > 0$$

and K^+, K^- are constants where $K^+ + K^- > 0$ in order to avoid infinite profit from cycling. Here, h models the running payoff resulting from the project's operation, and K^+ (resp., K^-) models the costs associated with increasing (resp., decreasing) the project's capacity level.

As it stands in (1.3), the performance index $J_{x,y}$ is not necessarily well-defined because the random variable inside the expectation may not be integrable or even well-defined. To address this issue, we define

$$U_T = \int_0^T e^{-rt} h(X_t, Y_t)\, dt - K^+ \int_{[0,T]} e^{-rt}\, d\xi_t^+ - K^- \int_{[0,T]} e^{-rt}\, d\xi_t^-, \quad \text{for } T \geq 0. \tag{1.4}$$

In the next section (see Lemma 4, in particular), we are going to impose assumptions on h such that U_T is well-defined, for all $T > 0$, and *either*

$$U_\infty = \lim_{T \to \infty} U_T \text{ exists in } \mathbb{R},\ P\text{-a.s.,} \quad \text{and} \quad U_\infty \in L^1(\Omega, \mathcal{F}, P), \tag{1.5}$$

in which case, we naturally define

$$J_{x,y}(\xi^+, \xi^-) = E\left[U_\infty\right], \tag{1.6}$$

as in (1.3), *or* there exists an $(\mathcal{F}_t)$-adapted process Z such that

$$U_T \le Z_T, \text{ for all } T \ge 0, \quad \text{and} \quad \limsup_{T\to\infty} E\left[Z_T\right] = -\infty, \tag{1.7}$$

in which case, we define

$$J_{x,y}(\xi^+, \xi^-) = -\infty. \tag{1.8}$$

The objective is to maximise the performance index $J_{x,y}$ thus defined over all admissible decision strategies $(\xi^+, \xi^-) \in \Pi_y$. The value function of the resulting optimisation problem is defined by

$$v(x, y) = \sup_{(\xi^+,\xi^-)\in\Pi_y} J_{x,y}(\xi^+, \xi^-). \tag{1.9}$$

1.3 Assumptions and preliminary estimates

The purpose of this section is to establish conditions on the problem's data under which our control problem is well-posed and its value function is finite, and to prove certain estimates that will be used in our analysis. Before we address these issues, we first discuss an ODE that will play an instrumental role in the solution of our control problem.

Let $k :]0, \infty[\to \mathbb{R}$ be any measurable function such that

$$E_x\left[\int_0^\infty e^{-rt} \left|k(X_t)\right| dt\right] < \infty, \quad \text{for all } x > 0. \tag{1.10}$$

In view of the results in Proposition 4.1 of Knudsen, Meister and Zervos [KMZ98], the function $R^{[k]} :]0, \infty[\to \mathbb{R}$ given by

$$R^{[k]}(x) = \frac{1}{\sigma^2(n-m)}\left[x^m \int_0^x s^{-m-1} k(s)\, ds + x^n \int_x^\infty s^{-n-1} k(s)\, ds\right] \tag{1.11}$$

is well-defined and

$$R^{[k]}(x) = E_x\left[\int_0^\infty e^{-rt} k(X_t)\, dt\right] \tag{1.12}$$

Here, the constants $m < 0 < n$ are the solutions of the quadratic equation

$$\sigma^2\lambda^2 + (b - \sigma^2)\lambda - r = 0, \tag{1.13}$$

given by

$$m, n = \frac{-(b-\sigma^2) \pm \sqrt{(b-\sigma^2)^2 + 4\sigma^2 r}}{2\sigma^2}. \tag{1.14}$$

Moreover, condition (1.10) is also sufficient for every solution of the ODE

$$\sigma^2 x^2 u''(x) + bxu'(x) - ru(x) + k(x) = 0 \tag{1.15}$$

to be expressed by

$$u(x) = Ax^n + Bx^m + R^{[k]}(x), \tag{1.16}$$

for some $A, B \in \mathbb{R}$.

With regard to $R^{[k]}$,

$$\text{if } k \text{ is increasing, then } R^{[k]} \text{ is increasing,} \tag{1.17}$$

and

$$\text{if } k \text{ is increasing, then } \inf_{x>0} \frac{k(x)}{r} \geq 0 \;\Leftrightarrow\; \inf_{x>0} R^{[k]}(x) \geq 0. \tag{1.18}$$

Also, for future reference, we note that, given any $\lambda \in \mathbb{R}$,

$$\begin{aligned} E\left[\int_0^\infty e^{-rt} X_t^\lambda \, dt\right] &= x^\lambda \int_0^\infty e^{[\sigma^2\lambda^2 + (b-\sigma^2)\lambda - r]t} E\left[e^{-\sigma^2\lambda^2 t + \sqrt{2}\sigma\lambda W_t}\right] dt \\ &= \begin{cases} \infty, & \text{if } \lambda \leq m \text{ or } \lambda \geq n, \\ -x^\lambda / \left[\sigma^2\lambda^2 + (b-\sigma^2)\lambda - r\right], & \text{if } \lambda \in]m, n[. \end{cases} \end{aligned} \tag{1.19}$$

We are going to need the following estimate that is related with the definitions above.

Lemma 1 *Given any $\lambda \in]0, n[$, there exist constants $\varepsilon_1, \varepsilon_2 > 0$ such that*

$$E\left[e^{-rt} \bar{X}_t^\lambda\right] \leq \frac{\sigma^2\lambda^2 + \varepsilon_2}{\varepsilon_2} x^\lambda e^{-\varepsilon_1 t} \quad \textit{and} \quad E\left[\sup_{t \geq 0} e^{-rt} \bar{X}_t^\lambda\right] \leq \frac{\sigma^2\lambda^2 + \varepsilon_2}{\varepsilon_2} x^\lambda,$$

where $\bar{X}_t = \sup_{s \leq t} X_s$.

Proof. Since n is the positive solution of the quadratic equation (1.13), it follows that there exist $\varepsilon_1, \varepsilon_2 > 0$ such that

$$r - \varepsilon_1 > 0 \quad \text{and} \quad \sigma^2\lambda^2 + (b - \sigma^2)\lambda - (r - \varepsilon_1) = -\varepsilon_2.$$

Given such parameters, we define

$$V = \sup_{t \geq 0} \left[-\frac{\sigma^2\lambda^2 + \varepsilon_2}{\sqrt{2}|\sigma|\lambda} t + W_t \right],$$

we calculate

$$\begin{aligned}
e^{-rt}\bar{X}_t^\lambda &= x^\lambda e^{-\varepsilon_1 t} e^{-(r-\varepsilon_1)t} \sup_{s \leq t} \exp\big((r - \varepsilon_1)s - (\sigma^2\lambda^2 + \varepsilon_2)s + \sqrt{2}\sigma\lambda W_s\big) \\
&= x^\lambda e^{-\varepsilon_1 t} \sup_{s \leq t} \left[\exp\big(-(r - \varepsilon_1)(t - s)\big) \exp\left(-(\sigma^2\lambda^2 + \varepsilon_2)s + \sqrt{2}\sigma\lambda W_s\right) \right] \\
&\leq x^\lambda e^{-\varepsilon_1 t} e^{\sqrt{2}|\sigma|\lambda V},
\end{aligned}$$

and we observe that

$$\sup_{t \geq 0} e^{-rt}\bar{X}_t^\lambda \leq x^\lambda e^{\sqrt{2}|\sigma|\lambda V}.$$

Since V is exponentially distributed with parameter $2\left(\sigma^2\lambda^2 + \varepsilon_2\right) / \left(\sqrt{2}|\sigma|\lambda\right)$ (see Karatzas and Shreve [KS88, Exercise 3.5.9]), the two bounds follow by a simple integration. □

The following assumptions on the data of the control problem formulated in Section 1.2 will ensure that the associated free-boundary problem has a unique solution that conforms with economical intuition.

Assumption 1 The function h is C^3, and, if we define

$$H(x, y) = h_y(x, y), \quad (x, y) \in \mathcal{S}, \tag{1.20}$$

then, given any $y > 0$,

$$H_x(x, y) > 0, \text{ for all } x > 0, \quad \text{and} \quad \lim_{x \to \infty} H(x, y) = \infty, \tag{1.21}$$

and, given any $x > 0$,

$$H_y(x, y) < 0, \quad \text{for all } y > 0. \tag{1.22}$$

Also, $K^+ + K^- > 0$, and

$$\int_0^x s^{-m-1} \left| H_y(s,y) \right| ds + \int_x^\infty s^{-n-1} \left| H_y(s,y) \right| ds < \infty, \quad \text{for all } y > 0.$$

□

It is worth observing that (1.21) and (1.22) in this assumption have a natural economic interpretation. Indeed, we can think of $H(x,y)\Delta y$ as the *additional* running payoff that we are faced with if we increase the project's capacity level from y to $y + \Delta y$, for small Δy, and the underlying state process X assumes the value x. In view of this observation, (1.21) reflects the idea that, given y, a small amount of extra capacity should be associated with increasing values of additional running payoff as the value of x, which, e.g., models the price of or the demand for the project's output commodity, is increasing. Similarly, (1.22) reflects the fact that, for a given value x of the underlying state process, the extra running payoff resulting from a small amount of additional capacity is decreasing as the level of the already installed capacity y increases. Also, the assumption that $K^+ + K^- > 0$, which is an indispensable one, is a most realistic one. Indeed, the inequality $K^+ + K^- < 0$ gives rise to the unrealistic scenario where the project's management can realise arbitrarily high profits by just sequentially increasing and then decreasing the project's capacity by the same amount sufficiently fast.

The following additional assumptions will ensure that the value function of the control problem considered is finite and identifies with the solution of the associated Hamilton-Jacobi-Bellman equation. Apart from (1.26), which can be justified by straightforward economics considerations such as the ones discussed above, the conditions in the assumption are of a technical nature.

Assumption 2 $K^+ > 0$, and there exist constants

$$\alpha > 0, \ \beta \in]0,1[, \ \vartheta \in]0, K^+ \wedge (K^+ + K^-) \wedge n[\text{ and } C > 0$$

where $n > 0$ is as in (1.14), such that

$$\frac{\alpha}{1-\beta} \in]0, n[, \tag{1.23}$$

$$-C(1+y) \le h(x,y) \le C(1 + x^{n-\vartheta} + x^\alpha y^\beta) + r(K^+ - \vartheta)y, \quad \text{for all } (x,y) \in \mathcal{S}. \tag{1.24}$$

$$-C \le H(x,y) \equiv h_y(x,y) \le \beta C x^\alpha y^{-(1-\beta)} + r(K^+ - \vartheta), \quad \text{for all } x, y > 0. \tag{1.25}$$

Also,

$$h_x(x,y) \ge 0, \quad \text{for all } y \ge 0. \tag{1.26}$$

□

Remark 1 Note that we could have replaced the upper bound in (1.25) by

$$H(x,y) \le \begin{cases} C\left(1 + x^\alpha y^{-(1-\beta)}\right), & \text{for all } x > 0 \text{ and } y < y_1, \\ \beta C x^\alpha y^{-(1-\beta)} + r(K^+ - \vartheta), & \text{for all } x > 0 \text{ and } y \ge y_1, \end{cases}$$

for some constant $y_1 > 0$. Depending on the problem's data, such a significant relaxation could result in optimal policies such as the one depicted by Figure 1.5 that would enrich qualitatively the class of optimal capacity control strategies. However, we decided against such a relaxation because this would complicate both the presentation and the analysis of our results.

Example 1 A choice for the running payoff function h that has been widely considered in the literature is the so-called Cobb-Douglas production function given by

$$h(x,y) = x^\alpha y^\beta, \quad \text{for some constants } \alpha > 0 \text{ and } \beta \in]0,1[. \tag{1.27}$$

A related choice that incorporates a running cost proportional to the project's installed capacity is given by

$$h(x,y) = x^\alpha y^\beta - Ky, \quad \text{for some constants } \alpha, K > 0 \text{ and } \beta \in]0,1[.$$

It is straightforward to verify that these choices for the running payoff function h satisfy all of our assumptions if and only if the parameters α and β as in (1.27) satisfy the inequality (1.23). □

It is a straightforward exercise to show that the bounds in (1.24)–(1.25) imply the following estimate.

Lemma 2 *With reference to the notation in (1.11), the bounds provided by (1.24) and (1.25) in Assumption 2 imply that there exists a constant $C_1 > 0$ such that*

$$-C_1(1+y) \le R^{[h(\cdot,y)]}(x) \le C_1\left(1+y+x^{n-\vartheta}+x^{\alpha}y^{\beta}\right), \quad \textit{for all } (x,y) \in \mathcal{S},$$
$$-C_1 \le R^{[H(\cdot,y)]}(x) \le C_1\left(1+x^{\alpha}y^{-(1-\beta)}\right), \quad \textit{for all } (x,y) \in \mathcal{S}.$$

As we have remarked above, bounds such as the ones in (1.24)–(1.25) are essential for the value function to be finite. Indeed, we can prove the following result.

Lemma 3 *Consider the control problem formulated in Section 1.2 that arises if the running payoff function h is defined by (1.27) in Example 1, and suppose that $\frac{\alpha}{1-\beta} > n > \alpha$. Then, under any well-posed definition of the performance index $J_{x,y}$ that is consistent with (1.3), $v(x,y) = \infty$, for every initial condition $(x,y) \in \mathcal{S}$.*

Proof. Consider the strategy defined by

$$\tilde{\xi}_t^+ = \bar{X}_t^{(n-\alpha)/\beta} \quad \text{and} \quad \tilde{\xi}_t^- = 0, \quad \text{for all } t \ge 0, \tag{1.28}$$

where $\bar{X}_t = \sup_{s\le t} X_s$. With regard to (1.19), we can see that this strategy is associated with

$$E\left[\int_0^\infty e^{-rt}X_t^\alpha \tilde{Y}_t^\beta\, dt\right] \ge E\left[\int_0^\infty e^{-rt}X_t^n\, dt\right] = \infty. \tag{1.29}$$

Now, let us assume that $\frac{\alpha}{1-\beta} > n > \alpha$. If we define $\lambda = \frac{n-\alpha}{\beta} > 0$, then such an assumption implies $\lambda < n$. In view of this observation, we can use the first estimate in Lemma 1, the monotone convergence theorem and the integration by parts formula to see that the strategy given by (1.28) satisfies

$$\begin{aligned}
E\left[\int_{[0,\infty[} e^{-rt}\, d\tilde{\xi}_t^+\right] &= \lim_{T\to\infty} E\left[r\int_0^T e^{-rt}\tilde{\xi}_t^+\, dt + e^{-rT}\tilde{\xi}_{T+}^+\right] \\
&= \lim_{T\to\infty}\left(r\int_0^T E\left[e^{-rt}\bar{X}_t^\lambda\right] dt + E\left[e^{-rT}\bar{X}_T^\lambda\right]\right) \\
&\le r\frac{\sigma^2\lambda^2+\varepsilon_2}{\varepsilon_1\varepsilon_2}x^\lambda \\
&< \infty.
\end{aligned}$$

However, this calculation, (1.28) and (1.29) imply that

$$E\left[\int_0^\infty e^{-rt}X_t^\alpha \tilde{Y}_t^\beta\,dt - \int_{[0,\infty[} e^{-rt}\,d\tilde{\xi}_t^+ - \int_{[0,\infty[} e^{-rt}\,d\tilde{\xi}_t^-\right]$$

is well-defined and equal to ∞, which proves the result. □

We can now prove that our assumptions are sufficient for the optimisation problem considered to be well-posed and for its value function to be finite.

Lemma 4 *Suppose that the running payoff function h satisfies (1.24) in Assumption 2 and that $K^+, K^+ + K^- > 0$. Given any initial condition $(x, y) \in \mathcal{S}$, (1.5)-(1.8) provide a well-posed definition of the performance criterion $J_{x,y}$, and the following statements hold true:*

(a) *Given any admissible strategy $(\xi^+, \xi^-) \in \Pi_y$, $J_{x,y}(\xi^+, \xi^-) \in \mathbb{R}$ if and only if*

$$E\left[\int_0^\infty e^{-rt}Y_t\,dt + \int_{[0,\infty[} e^{-rt}\,d\xi_t^+ + \int_{[0,\infty[} e^{-rt}\,d\xi_t^-\right] < \infty. \tag{1.30}$$

(b) *Condition (1.30) implies*

$$\liminf_{T\to\infty} e^{-rT}E\left[Y_{T+}\right] = 0. \tag{1.31}$$

(c) $v(x, y) \in \mathbb{R}$.

Proof. Fix any initial condition $(x, y) \in \mathcal{S}$ and any admissible strategy $(\xi^+, \xi^-) \in \Pi_y$. Since ξ^+, ξ^- are increasing càglàd processes with $\xi_0^+ = \xi_0^- = 0$, we can use the integration by parts formula to calculate

$$\begin{aligned}-K^+\int_{[0,T]} e^{-rt}\,d\xi_t^+ - K^-\int_{[0,T]} e^{-rt}\,d\xi_t^- \\ = -r\int_0^T e^{-rt}\left[K^+\xi_t^+ + K^-\xi_t^-\right]dt - e^{-rT}\left[K^+\xi_{T+}^+ + K^-\xi_{T+}^-\right].\end{aligned} \tag{1.32}$$

With regard to (1.1) and the inequality $K^+ + K^- > 0$, we can see that

$$-K^+\xi_t^+ - K^-\xi_t^- \le -K^+\left(\xi_t^+ - \xi_t^-\right) = -K^+Y_t + K^+y, \tag{1.33}$$

which, combined with (1.32), implies

$$-K^+\int_{[0,T]} e^{-rt}\,d\xi_t^+ - K^-\int_{[0,T]} e^{-rt}\,d\xi_t^- \le -rK^+\int_0^T e^{-rt}Y_t\,dt - e^{-rT}K^+Y_{T+} + K^+y. \tag{1.34}$$

However, this inequality and (1.24) in Assumption 2 imply that the random variables U_T defined by (1.4) satisfy

$$\begin{aligned} U_T &\le K^+ y + \int_0^T e^{-rt}\left[h(X_t, Y_t) - rK^+ Y_t\right] dt \\ &\le K^+ y + C\int_0^T e^{-rt}\left(1 + X_t^{n-\vartheta}\right) - \hat{Z}_T, \end{aligned} \tag{1.35}$$

where

$$\hat{Z}_T = \int_0^T e^{-rt}\left[r\vartheta Y_t - C X_t^{\alpha} Y_t^{\beta}\right] dt, \quad \text{for } T \ge 0.$$

With reference to (1.19),

$$\begin{aligned} I_1(x) &:= E\left[C\int_0^\infty e^{-rt}\left(1 + X_t^{n-\vartheta}\right) dt\right] \\ &= \frac{C}{r} - \frac{Cx^{n-\vartheta}}{\sigma^2(n-\vartheta)^2 + (b-\sigma^2)(n-\vartheta) - r} \in]0, \infty[. \end{aligned} \tag{1.36}$$

Now, suppose that the strategy $(\xi^+, \xi^-) \in \Pi_y$ is associated with

$$E\left[\int_0^\infty e^{-rt} Y_t\, dt\right] = \infty. \tag{1.37}$$

With regard to (1.23) in Assumption 2 and (1.19), we observe that

$$I_2(x) := E\left[\int_0^\infty e^{-rt} X_t^{\alpha/(1-\beta)}\, dt\right] < \infty. \tag{1.38}$$

Therefore, given any constant $\mu > 0$,

$$E\left[\int_0^\infty e^{-rt} Y_t \mathbf{1}_{\left\{Y_t < \mu X_t^{\alpha/(1-\beta)}\right\}}\, dt\right] \le \mu I_2(x) < \infty. \tag{1.39}$$

It follows that (1.37) is true if and only if

$$E\left[\int_0^\infty e^{-rt} Y_t \mathbf{1}_{\left\{Y_t \ge \mu X_t^{\alpha/(1-\beta)}\right\}}\, dt\right] = \infty. \tag{1.40}$$

Now, let any $\mu > 0$ such that $r\vartheta - C\mu^{-(1-\beta)} > 0$, where the constants $\vartheta, C > 0$ and $\beta \in]0,1[$ are as in Assumption 2, and note that

$$\begin{aligned} E\left[\hat{Z}_T\right] \ge & - C\mu^{\beta} E\left[\int_0^T e^{-rt} X_t^{\alpha/(1-\beta)} \mathbf{1}_{\left\{Y_t < \mu X_t^{\alpha/(1-\beta)}\right\}}\, dt\right] \\ & + \left(r\vartheta - C\mu^{-(1-\beta)}\right) E\left[\int_0^T e^{-rt} Y_t \mathbf{1}_{\left\{Y_t \ge \mu X_t^{\alpha/(1-\beta)}\right\}}\, dt\right]. \end{aligned}$$

In view of (1.39)–(1.40) and the monotone convergence theorem, the right hand side of this inequality tends to ∞ as $T \to \infty$, which implies that $\lim_{T\to\infty} E[\hat{Z}_T] = \infty$. However, this conclusion, (1.35) and (1.36) imply that there exists a process Z such that (1.7) is satisfied and, therefore, $J_{x,y}(\xi^+, \xi^-) = -\infty$.

To proceed further, let us assume that

$$E\left[\int_0^\infty e^{-rt} Y_t \, dt\right] < \infty, \tag{1.41}$$

which is necessary for condition (1.30) to be satisfied. Since Y is a finite variation process, its sample paths can have at most countable discontinuities. Using Fubini's theorem, we can see that this observation and (1.41) imply

$$\int_0^\infty e^{-rt} E\left[Y_{t+}\right] dt = E\left[\int_0^\infty e^{-rt} Y_{t+} \, dt\right] = E\left[\int_0^\infty e^{-rt} Y_t \, dt\right] < \infty,$$

which proves that (1.30) implies (1.31), and establishes part (b) of the lemma.

Now, using Hölder's inequality, we calculate

$$E\left[\int_0^\infty e^{-rt} X_t^\alpha Y_t^\beta \, dt\right] \leq I_2^{1-\beta}(x) \left(E\left[\int_0^\infty e^{-rt} Y_t \, dt\right]\right)^\beta < \infty, \tag{1.42}$$

where $I_2(x)$ is given by (1.38). This inequality, (1.36), (1.41) and the bounds in (1.24) in Assumption 2 imply

$$\begin{aligned} E\left[\int_0^\infty e^{-rt} \left|h(X_t, Y_t)\right| dt\right] &\leq E\left[\int_0^\infty e^{-rt} \left[C\left(1 + X_t^{n-\vartheta} + X_t^\alpha Y_t^\beta\right) + r(K^+ - \vartheta) Y_t\right] dt\right] \\ &< \infty, \end{aligned}$$

which combined with the dominated convergence theorem, implies that

$$\lim_{T\to\infty} E\left[\int_0^T e^{-rt} h(X_t, Y_t) \, dt\right] = E\left[\int_0^\infty e^{-rt} h(X_t, Y_t) \, dt\right] \in \mathbb{R}. \tag{1.43}$$

This observation gives rise to two possibilities. The first one is associated with the inequality

$$E\left[\int_{[0,\infty[} e^{-rt} \, d\xi_t^+ + \int_{[0,\infty[} e^{-rt} \, d\xi_t^-\right] < \infty.$$

In this case, $\lim_{T\to\infty} U_T$ exists, P-a.s., and belongs to $L^1(\Omega, \mathcal{F}, P)$, so $J_{x,y}(\xi^+, \xi^-)$ is finite and is given by (1.6). The second possibility is associated with

$$E\left[\int_{[0,\infty[} e^{-rt} \, d\xi_t^+ + \int_{[0,\infty[} e^{-rt} \, d\xi_t^-\right] = \infty,$$

which combined with (1.41) implies

$$E\left[\int_{[0,\infty[} e^{-rt}\,d\xi_t^+\right] = E\left[\int_{[0,\infty[} e^{-rt}\,d\xi_t^-\right] = \infty. \tag{1.44}$$

If $K^- < 0$, then we can use (1.1) and the integration by parts formula to calculate

$$\begin{aligned}
K^- \int_{[0,T]} e^{-rt}\,d\xi_t^- &= K^- \int_{[0,T]} e^{-rt}\,d\xi_t^+ + |K^-| \int_{[0,T]} e^{-rt}\,dY_t \\
&= K^- \int_{[0,T]} e^{-rt}\,d\xi_t^+ + r|K^-| \int_0^T e^{-rt} Y_t\,dt + |K^-| e^{-rT} Y_{T+} - |K^-| y \\
&\geq K^- \int_{[0,T]} e^{-rt}\,d\xi_t^+ - |K^-| y,
\end{aligned}$$

which implies

$$E\left[K^+ \int_{[0,T]} e^{-rt}\,d\xi_t^+ + K^- \int_{[0,T]} e^{-rt}\,d\xi_t^-\right] \geq (K^+ + K^-) E\left[\int_{[0,T]} e^{-rt}\,d\xi_t^+\right] - |K^-| y.$$

This inequality, the assumption that $K^+ + K^- > 0$, (1.44) and the monotone convergence theorem imply

$$\lim_{T\to\infty} E\left[K^+ \int_{[0,T]} e^{-rt}\,d\xi_t^+ + K^- \int_{[0,T]} e^{-rt}\,d\xi_t^-\right] = \infty \tag{1.45}$$

On the other hand, if $K^- \geq 0$, then (1.44) plainly implies (1.45). However, (1.43) and (1.45) imply that $\lim_{T\to\infty} E[U_T] = -\infty$, so (1.7) is satisfied for $Z = U$ and $J_{x,y}(\xi^+, \xi^-) = -\infty$.

The analysis above establishes the well-posedness of the definition of $J_{x,y}$ given by (1.5)–(1.8) as well as parts (a) and (b) of the lemma. To prove part (c) of the lemma, we first note that the results presented in (1.10)–(1.12) and the bounds in Lemma 2 imply

$$R^{[h(\cdot,y)]}(x) = E\left[\int_0^\infty e^{-rt} h(X_t, y)\,dt\right] \in \mathbb{R}.$$

However, this shows that our performance criterion is finite for the strategy that involves no capacity changes at any time, which proves that $v(x,y) > -\infty$. To show that $v(x,y) < \infty$, consider any admissible decision strategy $(\xi^+, \xi^-) \in \Pi_y$ such that

$J_{x,y}(\xi^+,\xi^-) > -\infty$. With reference to (1.41) and (1.42),

$$
\begin{aligned}
E\left[\int_0^\infty e^{-rt}\left[r\vartheta Y_t - CX_t^\alpha Y_t^\beta\right] dt\right]& \\
&\geq r\vartheta E\left[\int_0^\infty e^{-rt}Y_t\, dt\right] - CI_2^{1-\beta}(x)\left(E\left[\int_0^\infty e^{-rt}Y_t\, dt\right]\right)^\beta \\
&\geq -\frac{(1-\beta)r\vartheta}{\beta}\left(\frac{\beta C}{r\vartheta}\right)^{1/(1-\beta)} I_2(x), \quad \text{for all } T>0,
\end{aligned} \tag{1.46}
$$

the second inequality following because, given any constants $\kappa, \lambda > 0$ and $\beta \in]0,1[$,

$$
\kappa Q - \lambda Q^\beta \geq -\frac{(1-\beta)\kappa}{\beta}\left(\frac{\beta\lambda}{\kappa}\right)^{1/(1-\beta)}, \quad \text{for all } Q \geq 0,
$$

in particular, for $Q = E\left[\int_0^\infty e^{-rt}Y_t\, dt\right]$. However, (1.35), (1.36) and (1.46) imply

$$
J_{x,y}(\xi^+,\xi^-) \leq I_1(x) + K^+y + \frac{(1-\beta)r\vartheta}{\beta}\left(\frac{\beta C}{r\vartheta}\right)^{1/(1-\beta)} I_2(x),
$$

which proves that $v(x,y) < \infty$ because the right hand side of this inequality is finite and independent of ξ^+ and ξ^-. □

1.4 The Hamilton-Jacobi-Bellman (HJB) equation

The problem described in the previous section has the structure of a singular stochastic control problem. With regard to standard theory of singular control, we expect that its value function can be identified with a solution $w : \mathcal{S} \to \mathbb{R}$ to the HJB quasi-variational inequalities

$$
\begin{aligned}
\max\{\sigma^2x^2w_{xx}(x,y) + bxw_x(x,y) - rw(x,y) + h(x,y),& \\
w_y(x,y) - K^+,\ -w_y(x,y) - K^-\} = 0,& \quad x,y>0,
\end{aligned} \tag{1.47}
$$

$$
\max\{\sigma^2x^2w_{xx}(x,0) + bxw_x(x,0) - rw(x,0) + h(x,0),\ w_y(x,0) - K^+\} = 0, \quad x>0, \tag{1.48}
$$

where $w_y(x,0) := \lim_{y\downarrow 0} w_y(x,y)$.

To obtain some qualitative understanding of the origins of this equation, we observe that, at time 0, the project's management has to choose between three options. The first one is to wait for a short time Δt, and then continue optimally. With respect to Bellman's principle of optimality, this option is associated with the inequality

$$v(x,y) \geq E\left[\int_0^{\Delta t} e^{-rt} h(X_t, y)\,dt + e^{-r\Delta t} v(X_{\Delta t}, y)\right].$$

Applying Itô's formula to the second term in the expectation, and dividing by Δt before letting $\Delta t \downarrow 0$, we obtain

$$\sigma^2 x^2 v_{xx}(x,y) + bxv_x(x,y) - rv(x,y) + h(x,y) \leq 0. \tag{1.49}$$

The second option is to increase capacity immediately by $\varepsilon > 0$, and then continue optimally. This action is associated with the inequality

$$v(x,y) \geq v(x, y+\varepsilon) - K^+\varepsilon.$$

Rearranging terms and letting $\varepsilon \downarrow 0$, we obtain

$$v_y(x,y) - K^+ \leq 0. \tag{1.50}$$

Assuming that $y > 0$, the final option is to decrease capacity immediately by $\varepsilon > 0$, and then continue optimally. This option yields the inequality

$$v(x,y) \geq v(x, y-\varepsilon) - K^-\varepsilon,$$

which in the limit $\varepsilon \downarrow 0$ implies

$$-v_y(x,y) - K^- \leq 0. \tag{1.51}$$

Since these three are the only options available, we expect that one of them should be optimal, so that one of the inequalities (1.49)–(1.51) should hold with equality if $y > 0$, while, one of the inequalities (1.49)–(1.50) should hold with equality if $y = 0$. However, this observation combined with (1.49)–(1.51) implies that the value function v should identify with a solution w to (1.47)–(1.48).

The following result is concerned with *sufficient* conditions under which the value function v of the control problem considered identifies with a solution to (1.47)–(1.48). We impose some of these conditions, (1.56)–(1.57) in particular, which are not standard in similar "verification" theorems where usually the transversality condition is assumed. These inequalities imply the transversality condition and are used throughout the proof of our verification theorem as well as in our analysis in the next section.

Theorem 5 *Suppose that Assumption 2 holds and that the HJB equation (1.47)-(1.48) has a C^2 solution $w : \mathcal{S} \to \mathbb{R}$ such that*

$$-C_2\left(1 + y + x^{\alpha/(1-\beta)}\right) \leq w(x,y), \quad \textit{for all } (x,y) \in \mathcal{S}, \tag{1.52}$$

for some constant $C_2 > 0$. The following statements hold true:

(a) *$v(x,y) \leq w(x,y)$, for all initial conditions $(x,y) \in \mathcal{S}$.*

(b) *Given any initial condition $(x,y) \in \mathcal{S}$, suppose that there exists a decision strategy $(\xi^{o+}, \xi^{o-}) \in \Pi_y$ such that, if Y^o is the associated capacity process, then*

$$(X_t, Y_t^o) \in \left\{(x,y) \in \mathcal{S} : \ \sigma^2 x^2 w_{xx}(x,y) + bxw_x(x,y) - rw(x,y) + h(x,y) = 0\right\}, \tag{1.53}$$

Lebesgue-a.e., P-a.s.,

$$\int_{[0,T]} e^{-rs}\left[w_y(X_t, Y_t) - K^+\right] d\xi_s^{o+} = 0, \quad \textit{for all } T \geq 0, \ P\textit{-a.s.}, \tag{1.54}$$

$$\int_{[0,T]} e^{-rs}\left[w_y(X_t, Y_t) + K^-\right] d\xi_s^{o-} = 0, \quad \textit{for all } T \geq 0, \ P\textit{-a.s.}, \tag{1.55}$$

and

$$Y_t^o + X_t^\alpha (Y_t^o)^\beta + \xi_t^{o+} \leq C_3(y)\left(1 + \bar{X}_t^{n-\varepsilon_3}\right), \quad \textit{for all } t \geq 0, \ P\textit{-a.s.}, \tag{1.56}$$

$$w(X_t, Y_t^o) \leq C_3(y)\left(1 + \bar{X}_t^{n-\varepsilon_3}\right), \quad \textit{for all } t \geq 0, \ P\textit{-a.s.}, \tag{1.57}$$

where $\bar{X}_t = \sup_{s \leq t} X_s$, $\varepsilon_3 \in]0, \vartheta[$ is a constant, and $C_3(y) > 0$ is a constant depending on the initial condition y only. Then $v(x,y) = w(x,y)$ and (ξ^{o+}, ξ^{o-}) is the optimal strategy.

Proof. (a) Fix any initial condition (x,y) and any admissible strategy $(\xi^+, \xi^-) \in \Pi_y$ such that $J_{x,y}(\xi^+, \xi^-) > -\infty$, so that $J_{x,y}(\xi^+, \xi^-) = E[U_\infty]$ (see (1.5)-(1.6)). Using Itô's formula and the fact that X has continuous sample paths, we obtain

$$\begin{aligned} e^{-rT} w(X_T, Y_{T+}) = w(x,y) &+ \int_0^T e^{-rt}\left[\sigma^2 X_t^2 w_{xx}(X_t, Y_t) + bX_t w_x(X_t, Y_t) - rw(X_t, Y_t)\right] dt \\ &+ \int_{[0,T]} e^{-rt}\left[w_y(X_t, Y_t)\, d\xi_t^+ - w_y(X_t, Y_t)\, d\xi_t^-\right] + M_T \\ &+ \sum_{0 \leq t \leq T} e^{-rt}\left[w(X_t, Y_{t+}) - w(X_t, Y_t) - w_y(X_t, Y_{t-})\Delta Y_t\right], \end{aligned}$$

where

$$M_T = \sqrt{2}\sigma \int_0^T e^{-rt} X_t w_x(X_t, Y_t)\, dW_t, \quad T \geq 0. \tag{1.58}$$

Recalling the definition of U_T in (1.4), this implies

$$\begin{aligned}
&U_T + e^{-rT} w(X_T, Y_{T+}) \\
&\quad = w(x,y) + \int_0^T e^{-rt} \left[\sigma^2 X_t^2 w_{xx}(X_t, Y_t) + bX_t w_x(X_t, Y_t) - rw(X_t, Y_t) + h(X_t, Y_t)\right] dt \\
&\qquad + \int_{[0,T]} e^{-rt} \left[w_y(X_t, Y_t) - K^+\right] d\left(\xi^+\right)_t^{\mathrm{c}} + \int_{[0,T]} e^{-rt} \left[-w_y(X_t, Y_t) - K^-\right] d\left(\xi^-\right)_t^{\mathrm{c}} \\
&\qquad + M_T + \sum_{0 \leq t \leq T} e^{-rt} \left[w(X_t, Y_{t+}) - w(X_t, Y_t) - K^+ \Delta Y_t\right] \mathbf{1}_{\{\Delta Y_t > 0\}} \\
&\qquad + \sum_{0 \leq t \leq T} e^{-rt} \left[w(X_t, Y_{t+}) - w(X_t, Y_t) + K^- \Delta Y_t\right] \mathbf{1}_{\{\Delta Y_t < 0\}}.
\end{aligned}$$

Observing that

$$\left[w(X_t, Y_{t+}) - w(X_t, Y_t) - K^+ \Delta Y_t\right] \mathbf{1}_{\{\Delta Y_t > 0\}} = \mathbf{1}_{\{\Delta Y_t > 0\}} \int_0^{\Delta Y_t} \left[w_y(X_t, Y_t + u) - K^+\right] du,$$

$$\begin{aligned}
&\left[w(X_t, Y_{t+}) - w(X_t, Y_t) + K^- \Delta Y_t\right] \mathbf{1}_{\{\Delta Y_t < 0\}} \\
&\qquad\qquad = \mathbf{1}_{\{\Delta Y_t < 0\}} \int_0^{|\Delta Y_t|} \left[-w_y\left(X_t, Y_t - |\Delta Y_t| + u\right) - K^-\right] du,
\end{aligned}$$

we can see that, since w satisfies the HJB equation (1.47)–(1.48),

$$U_T + e^{-rT} w(X_T, Y_{T+}) \leq w(x,y) + M_T. \tag{1.59}$$

Now, in view of (1.34) and the assumption $K^+ > 0$,

$$-e^{-rT} Y_{T+} \geq -\int_{[0,T]} e^{-rt}\, d\xi_t^+ - \frac{|K^-|}{K^+} \int_{[0,T]} e^{-rt}\, d\xi_t^- - y,$$

which, combined with assumption (1.52), implies

$$e^{-rT} w(X_T, Y_{T+}) \geq -C_{21} \left(1 + \int_{[0,T]} e^{-rt}\, d\xi_t^+ + \int_{[0,T]} e^{-rt}\, d\xi_t^- + e^{-rT} X_T^{\alpha/(1-\beta)}\right),$$

for some constant $C_{21} = C_{21}(y) > 0$. Combining this inequality with

$$\int_0^T e^{-rt} h(X_t, Y_t)\, dt \geq -C \int_0^T e^{-rt} Y_t\, dt - \frac{C}{r}\left(1 - e^{-rT}\right),$$

which follows from (1.24) in Assumption 2, we can see that (1.59) implies

$$\inf_{T\geq 0} M_T \geq -C_{22}\left(1+\int_0^\infty e^{-rt}Y_t\,dt+\int_{[0,\infty[} e^{-rt}\,d\xi_t^+ +\int_{[0,\infty[} e^{-rt}\,d\xi_t^- +\sup_{T\geq 0} e^{-rT}\bar{X}_T^{\alpha/(1-\beta)}\right),$$

where $C_{22}=C_{22}(x,y)>0$ is a constant and $\bar{X}_t=\sup_{s\leq t}X_s$. Recalling the assumption that $\frac{\alpha}{1-\beta}\in\,]0,n[$, we can see that the second bound in Lemma 1 and (1.30) in Lemma 4 imply that the random variable on the right hand side of this inequality has finite expectation. It follows that the stochastic integral M defined by (1.58) is a supermartingale, and therefore, $E\left[M_T\right]\leq 0$, for all $T>0$. Taking expectations in (1.59), we therefore obtain

$$E\left[U_T\right]\leq w(x,y)+e^{-rT}E\left[-w(X_T,Y_{T+})\right]. \tag{1.60}$$

Furthermore, since

$$U_T\geq -C_{22}\left(1+\int_0^\infty e^{-rt}Y_t\,dt+\int_{[0,\infty[} e^{-rt}\,d\xi_t^+ +\int_{[0,\infty[} e^{-rt}\,d\xi_t^-\right),\quad \text{for all } T\geq 0,$$

and the random variable on the right hand side of this inequality has finite expectation, Fatou's lemma implies

$$J_{x,y}(\xi^+,\xi^-)\leq \liminf_{T\to\infty} E\left[U_T\right], \tag{1.61}$$

while (1.52) implies

$$\begin{aligned}\liminf_{T\to\infty} e^{-rT}E\left[-w(X_T,Y_{T+})\right] &\leq \lim_{T\to\infty} e^{-rT}C_2+C_2\liminf_{T\to\infty} e^{-rT}E\left[Y_{T+}\right]\\ &\quad +C_2\lim_{T\to\infty} e^{-rT}E\left[\bar{X}_T^{\alpha/(1-\beta)}\right]\\ &=0, \end{aligned}\tag{1.62}$$

the equality being true thanks to the first bound in Lemma 1 and (1.31). However, (1.60)–(1.62) imply that $J_{x,y}(\xi^+,\xi^-)\leq w(x,y)$, which establishes part (a) of the theorem.

(b) If (ξ^{o+},ξ^{o-}) is as in the statement of the theorem, then we can see that the monotone convergence theorem, the integration by parts formula, (1.56) and the first estimate in Lemma 1 imply

$$
\begin{aligned}
&E\left[\int_0^\infty e^{-rt} Y_t^\circ \, dt + \int_{[0,\infty[} e^{-rt} \, d\xi_t^\circ\right] \\
&\qquad = \lim_{T\to\infty} E\left[\int_0^T e^{-rt} Y_t^\circ \, dt + r\int_{[0,T]} e^{-rt}\xi_t^{\circ+} \, dt + e^{-rT}\xi_T^{\circ+}\right] \\
&\qquad \le (1+r)C_3(y)\left(\frac{1}{r} + \int_0^\infty e^{-rt} E\left[\bar{X}_t^{n-\varepsilon_3}\right] dt\right) + \lim_{T\to\infty} e^{-rT} E\left[\bar{X}_T^{n-\varepsilon_3}\right] \\
&\qquad < \infty,
\end{aligned}
$$

which, combined with (1.1), implies that (1.30) in Lemma 4 is satisfied, and, therefore,

$$
J_{x,y}(\xi^{\circ+}, \xi^{\circ-}) = E\left[\lim_{T\to\infty} U_T^\circ\right], \tag{1.63}
$$

where U° is defined as in (1.4). Furthermore, we can verify that (1.59) holds with equality, i.e.,

$$
U_T^\circ + e^{-rT} w(X_T, Y_{T+}^\circ) = w(x,y) + M_T^\circ, \tag{1.64}
$$

where the stochastic integral M° is defined as in (1.58). In view of (1.24) in Assumption 2 and (1.56), there exist constants $C_{31} > 0$ and $C_{32} = C_{32}(y) > 0$ such that

$$
\begin{aligned}
\sup_{T\ge 0} \int_0^T e^{-rt} h(X_t, Y_t^\circ) \, dt &\le C_{31}\left(1 + \int_0^\infty e^{-rt}\left[X_t^{n-\vartheta} + X_t^\alpha (Y_t^\circ)^\beta + Y_t^\circ\right] dt\right) \\
&\le C_{32}\left(1 + \int_0^\infty e^{-rt} \bar{X}_t^{n-\varepsilon_3} \, dt\right).
\end{aligned} \tag{1.65}
$$

With reference to (1.1), the assumption $K^+ + K^- > 0$, the integration by parts formula and (1.56), we can see that there exists a constant $C_{33} = C_{33}(y) > 0$ such that

$$
\begin{aligned}
&\sup_{T\ge 0}\left(-K^+ \int_{[0,T]} e^{-rt} \, d\xi_t^{\circ+} - K^- \int_{[0,T]} e^{-rt} \, d\xi_t^{\circ-}\right) \\
&\qquad \le \sup_{T\ge 0} K^-\left(\int_{[0,T]} e^{-rt} \, d\xi_t^{\circ+} - \int_{[0,T]} e^{-rt} \, d\xi_t^{\circ-}\right) \\
&\qquad \le |K^-| \sup_{T\ge 0} \int_{[0,T]} e^{-rt} \, dY_t^\circ \\
&\qquad \le |K^-| \sup_{T\ge 0} e^{-rT} Y_T^\circ + r|K^-| \int_0^\infty e^{-rt} Y_t^\circ \, dt \\
&\qquad \le |K^-| \sup_{T\ge 0} e^{-rT} Y_T^\circ + C_{33}\left(1 + \int_0^\infty e^{-rt} \bar{X}_t^{n-\varepsilon_3} \, dt\right).
\end{aligned} \tag{1.66}
$$

Moreover, (1.56)–(1.57) imply

$$\sup_{T\geq 0} e^{-rT} Y_T^{\circ} + \sup_{T\geq 0} e^{-rT} w(X_T, Y_T^{\circ}) \leq 2C_3(y) \left(1 + \sup_{T\geq 0} e^{-rT} \bar{X}_T^{n-\varepsilon_3}\right). \tag{1.67}$$

Now, (1.19) implies

$$E\left[\int_0^{\infty} e^{-rt} \bar{X}_t^{n-\varepsilon_3}\, dt\right] < \infty, \tag{1.68}$$

while the second estimate in Lemma 1 implies

$$E\left[\sup_{T\geq 0} e^{-rT} \bar{X}_T^{n-\varepsilon_3}\right] < \infty. \tag{1.69}$$

However, (1.64) and the estimates (1.65)–(1.69) imply that $E\left[\sup_{T\geq 0} M_T^{\circ}\right] < \infty$, which proves that the stochastic integral M° is a submartingale. Taking expectations in (1.64), we therefore obtain

$$E\left[U_T^{\circ}\right] \geq w(x,y) + e^{-rT} E\left[-w(X_T, Y_T^{\circ})\right]. \tag{1.70}$$

Furthermore, the estimates (1.65)–(1.69) imply that the random variables U_T°, indexed by $T \geq 0$, are all bounded from above by a random variable with finite expectation. This observation, (1.63) and Fatou's lemma imply

$$J_{x,y}(\xi^{\circ+}, \xi^{\circ-}) \geq \limsup_{T\to\infty} E\left[U_T^{\circ}\right]. \tag{1.71}$$

Finally, (1.57) and the first estimate in Lemma 1 imply

$$\begin{aligned}\limsup_{T\to\infty} e^{-rT} E\left[-w(X_T, Y_T^{\circ})\right] &\geq -\lim_{T\to\infty} C_3(y) \left(e^{-rT} + E\left[e^{-rT} \bar{X}_T^{n-\varepsilon_3}\right]\right) \\ &= 0,\end{aligned}$$

which, combined with (1.70) and (1.71), implies $J_{x,y}(\xi^{\circ+}, \xi^{\circ-}) \geq w(x,y)$. However, this inequality and part (a) of this theorem complete the proof. □

1.5 The solution of the control problem

We can now derive an explicit solution to the control problem formulated in Section 1.2 by constructing an appropriate solution w to the HJB equation (1.47)–(1.48). With respect to the heuristic arguments in Section 1.4 that led to the derivation of this equation, we start by conjecturing that the optimal strategy is characterised by three regions: the "wait" region $\mathcal{W}$ where (1.49) holds with equality, the "investment" region $\mathcal{I}$ where (1.50) holds with equality, and the "disinvestment" region $\mathcal{D}$ where (1.51) holds with equality. Also, we conjecture that each of the regions $\mathcal{W}$, $\mathcal{I}$, $\mathcal{D}$ is connected. In particular, we expect that, depending on the problem's data, the optimal strategy can take any of the forms depicted by Figures 1.1–1.4. Note that one can envisage other possibilities such as the one depicted by Figure 1.5. However, our assumptions do not allow for the optimality of such cases under any admissible choice of the problem's data (see also Remark 1).

With regard to Figures 1.1–1.4 on pages 40-42 at the end of this section, we denote by $\mathbb{F}$ and $\mathbb{G}$ the boundaries separating the regions $\mathcal{D}$, $\mathcal{W}$ and $\mathcal{W}$, $\mathcal{I}$, respectively, so that

$$\mathbb{F} = \overline{\mathcal{D}} \cap \overline{\mathcal{W}} \quad \text{and} \quad \mathbb{G} = \overline{\mathcal{W}} \cap \overline{\mathcal{I}},$$

where $\overline{\mathcal{W}}$, $\overline{\mathcal{I}}$ and $\overline{\mathcal{D}}$ are the closures of $\mathcal{W}$, $\mathcal{I}$ and $\mathcal{D}$ in $\mathbb{R}^2_+$, respectively. Furthermore, we define

$$y^* = \inf \{ y \geq 0 : \text{ there exists } x > 0 \text{ such that } (x, y) \in \mathbb{F} \}, \tag{1.72}$$

with the usual convention that $\inf \emptyset = \infty$. We will prove that

$$\begin{gathered} \text{there exists an increasing function } G : [0, \infty[\to [0, \infty[\text{ such that} \\ \mathbb{G} = \{ (G(y), y) : \ y \geq 0 \}, \end{gathered} \tag{1.73}$$

and, if $y^* < \infty$, then

$$\begin{gathered} \text{there exists an increasing function } F : [y^*, \infty[\to [0, \infty[\text{ such that} \\ \mathbb{F} \cap (\mathbb{R}_+ \setminus \{0\})^2 = \{ (F(y), y) : \ y > y^* \}. \end{gathered} \tag{1.74}$$

Given such a characterisation of $\mathbb{F}$ and $\mathbb{G}$,

$$\begin{aligned} \overline{\mathcal{W}} = & \left\{ (x, y) \in \mathbb{R}^2_+ : \ y \leq y^* \text{ and } x \in [0, G(y)] \right\} \\ & \cup \left\{ (x, y) \in \mathbb{R}^2_+ : \ y > y^* \text{ and } x \in [F(y), G(y)] \right\}, \\ \overline{\mathcal{I}} = & \left\{ (x, y) \in \mathbb{R}^2_+ : \ G(y) \leq x \right\}, \end{aligned}$$

while, if $y^* < \infty$, then

$$\mathcal{D} = \{(x,y) \in]0,\infty[\times \mathbb{R}_+ : \ y > y^* \text{ and } x \in]0, F(y)]\}.$$

In view of this structure, it is worth noting that, if $y^* = 0$ and $0 < F(0) < G(0)$ (see Figure 1.3), then $\{(x,0) : \ x < G(0)\} \subset \mathcal{W}$, so that the segment $]0, F(0)]$ is part of the "wait" region $\mathcal{W}$.

Inside the region $\mathcal{W}$, w satisfies the differential equation

$$\sigma^2 x^2 w_{xx}(x,y) + bxw_x(x,y) - rw(x,y) + h(x,y) = 0. \tag{1.75}$$

With regard to the discussion regarding the solvability of (1.15) in Section 1.3, every solution to this equation is given by

$$w(x,y) = A(y)x^n + B(y)x^m + R(x,y), \tag{1.76}$$

for some functions A and B. Here, the constants $m < 0 < n$ are given by (1.14), while the function $R \equiv R^{[h(\cdot,y)]}$ is given by

$$R(x,y) = \frac{1}{\sigma^2(n-m)}\left[x^m \int_0^x s^{-m-1}h(s,y)\,ds + x^n \int_x^\infty s^{-n-1}h(s,y)\,ds\right]. \tag{1.77}$$

For $y \in [0, y^*] \cap \mathbb{R}$, we must have $B(y) = 0$. This choice is supported by the heuristic observation that, for fixed capacity level $y \geq 0$, the problem's value function should remain bounded as the value x of the underlying state process tends to 0. Also, it eventually turns out that (1.56)–(1.57) in the verification Theorem 5 cannot be satisfied if $B(y) \neq 0$. To determine $A(y)$ and $G(y)$ when $y \in [0, y^*] \cap \mathbb{R}$, we postulate that $w(\cdot, y)$ is C^2 at the free-boundary point $G(y)$. In particular, we postulate that

$$\lim_{x \uparrow G(y)} w_y(x,y) = \lim_{x \downarrow G(y)} w_y(x,y) \quad \text{and} \quad \lim_{x \uparrow G(y)} w_{yx}(x,y) = \lim_{x \downarrow G(y)} w_{yx}(x,y). \tag{1.78}$$

Since w satisfies

$$w_y(x,y) = K^+, \quad \text{for } (x,y) \in \mathcal{I}, \tag{1.79}$$

which implies

$$w_{xy}(x,y) = 0, \quad \text{for } (x,y) \in \mathcal{I}, \tag{1.80}$$

this requirement yields the system of equations

$$A'(y)G^n(y) = K^+ - R_y(G(y), y), \tag{1.81}$$

$$A'(y)G^n(y) = -\frac{1}{n}G(y)R_{xy}(G(y), y). \tag{1.82}$$

With regard to (1.77) and the identity $\sigma^2 mn = -r$, this system implies that $G(y)$ should satisfy

$$q(G(y), y) = 0, \tag{1.83}$$

where

$$q(x, y) = \int_0^x s^{-m-1}\left[H(s, y) - rK^+\right] ds, \quad (x, y) \in \mathcal{S}, \tag{1.84}$$

and H is the function defined by (1.20). Also, we can calculate

$$\begin{aligned} A'(y) &= \frac{1}{2}G^{-n}(y)\left[K^+ - R_y(G(y), y) - \frac{1}{n}G(y)R_{xy}(G(y), y)\right] \\ &= -\frac{1}{\sigma^2(n-m)}\int_{G(y)}^{\infty} s^{-n-1}\left[H(s, y) - rK^+\right] ds. \end{aligned} \tag{1.85}$$

The following result is concerned with the solvability of equation (1.83).

Lemma 6 *Suppose that Assumption 1 is true. Given any $y \geq 0$, the equation $q(x, y) = 0$ has a unique solution $x = x(y) > 0$ if and only if $\inf_{x>0} H(x, y) < rK^+$. If we define*

$$\tilde{y}_* = \inf\left\{y \geq 0 : \inf_{x>0} H(x, y) < rK^+\right\}, \tag{1.86}$$

then equation (1.83) defines uniquely a function $\tilde{G} :]\tilde{y}_*, \infty[\to]0, \infty[$ that is C^1, strictly increasing, and satisfies

$$H(\tilde{G}(y), y) - rK^+ > 0, \quad \text{for all } y > \tilde{y}_*. \tag{1.87}$$

Furthermore, if (1.25) in Assumption 2 is also true, then $\tilde{y}_* = 0$ and

$$C_4^{-\frac{1-\beta}{\alpha}} y^{\frac{1-\beta}{\alpha}} \leq \tilde{G}(y), \text{ for all } y \geq 0 \Leftrightarrow \tilde{G}^{[-1]}(x) \leq C_4 x^{\frac{\alpha}{1-\beta}}, \text{ for all } x \geq \tilde{G}(0), \tag{1.88}$$

where $\tilde{G}^{[-1]}$ is the inverse function of $\tilde{G}$, $\tilde{G}(0) := \lim_{y \downarrow 0} \tilde{G}(y)$ and $C_4 > 0$ is a constant.

We collect in the Appendix the proofs of those results that are not developed in the text.

Now, let us consider the case where $\mathcal{D} \neq \emptyset$ and the point y^* defined by (1.72) is finite (see Figures 1.2–1.4). For $y > y^*$, w is given by (1.75) for x such that $(x, y) \in \mathcal{W}$, by (1.79) for x such that $(x, y) \in \mathcal{I}$, and by

$$w_y(x, y) = -K^-, \tag{1.89}$$

for x such that $(x, y) \in \mathcal{D}$. Plainly, C^2 continuity of w inside $\mathcal{D}$ implies

$$w_{xy}(x, y) = 0, \quad \text{for } (x, y) \in \mathcal{D}. \tag{1.90}$$

To determine $A(y)$, $B(y)$, $F(y)$ and $G(y)$, we postulate that $w(\cdot, y)$ is C^2 at both of the free-boundary points $F(y)$ and $G(y)$. With regard to (1.76), (1.79)–(1.80), (1.89)–(1.90), the definition (1.77) of $R(x, y)$ and the identity $\sigma^2 mn = -r$, this requirement yields

$$A'(y) = -\frac{1}{\sigma^2(n-m)} \int_{F(y)}^{\infty} s^{-n-1} \left[H(s, y) + rK^-\right] ds, \tag{1.91}$$

$$A'(y) = -\frac{1}{\sigma^2(n-m)} \int_{G(y)}^{\infty} s^{-n-1} \left[H(s, y) - rK^+\right] ds, \tag{1.92}$$

$$B'(y) = -\frac{1}{\sigma^2(n-m)} \int_{0}^{F(y)} s^{-m-1} \left[H(s, y) + rK^-\right] ds, \tag{1.93}$$

$$B'(y) = -\frac{1}{\sigma^2(n-m)} \int_{0}^{G(y)} s^{-m-1} \left[H(s, y) - rK^+\right] ds, \tag{1.94}$$

where H is defined by (1.20). These calculations imply that the points $F(y)$ and $G(y)$ should satisfy the system of equations

$$f(F(y), G(y), y) = 0, \tag{1.95}$$

$$g(F(y), G(y), y) = 0, \tag{1.96}$$

where

$$f(x_1, x_2, y) = \int_0^{x_1} s^{-m-1} \left[H(s, y) + rK^-\right] ds - \int_0^{x_2} s^{-m-1} \left[H(s, y) - rK^+\right] ds, \tag{1.97}$$

$$g(x_1, x_2, y) = \int_{x_1}^{\infty} s^{-n-1} \left[H(s, y) + rK^-\right] ds - \int_{x_2}^{\infty} s^{-n-1} \left[H(s, y) - rK^+\right] ds. \tag{1.98}$$

The following result is concerned with the solvability of the system of equations (1.95) and (1.96).

Lemma 7 *Suppose that Assumption 1 holds. Given $y \geq 0$, the system of equations (1.95) and (1.96) has a unique solution $(x_1, x_2) = (x_1(y), x_2(y))$ such that $0 < x_1 < x_2$ if and only if $\inf_{x>0} H(x,y) < -rK^-$. Moreover, if we define*

$$\bar{y}^* = \inf\left\{ y \geq 0 : \inf_{x>0} H(x,y) + rK^- < 0 \right\}, \tag{1.99}$$

with the usual convention that $\inf \emptyset = \infty$, then, if $\bar{y}^* < \infty$, the system of equations (1.95) and (1.96) defines uniquely two functions $\bar{F}, \bar{G} :]\bar{y}^*, \infty[\to]0, \infty[$ that are C^1, strictly increasing, and satisfy $\bar{F}(y) < \bar{G}(y)$, for all $y > \bar{y}^*$,

$$\bar{F}(\bar{y}^*) := \lim_{y \downarrow \bar{y}^*} \bar{F}(y) = 0, \quad \text{if } \bar{y}^* > 0, \tag{1.100}$$

$$\bar{F}(0) := \lim_{y \downarrow 0} \bar{F}(y) \leq \lim_{y \downarrow 0} \bar{G}(y) =: \bar{G}(0), \quad \text{if } \bar{y}^* = 0, \tag{1.101}$$

$$H(\bar{F}(y), y) + rK^- < 0 \quad \text{and} \quad H(\bar{G}(y), y) - rK^+ > 0, \quad \text{for all } y > \bar{y}^*. \tag{1.102}$$

Furthermore, if (1.25) in Assumption 2 also holds, then

$$C_4^{-\frac{1-\beta}{\alpha}} y^{\frac{1-\beta}{\alpha}} \leq \bar{G}(y), \text{ for all } y \geq \bar{y}^* \Leftrightarrow \bar{G}^{[-1]}(x) \leq C_4 x^{\frac{\alpha}{1-\beta}}, \text{ for all } x \geq \bar{G}(\bar{y}^*) \tag{1.103}$$

where $\bar{G}^{[-1]}$ is the inverse function of $\bar{G}$ and the constant $C_4 > 0$ is the same constant as in Lemma 6.

In light of the results above, and in the presence of (1.25) in Assumption 2, the point $\bar{y}^*$ defined by (1.99) identifies with the point y^* in (1.72), while, the functions $F : [y^*, \infty[\to [0, \infty[$ and $G : [0, \infty[\to [0, \infty[$ separating the three possible regions, as conjectured in (1.73)–(1.74), are given by

$$F = \bar{F}, \text{ if } y^* < \infty, \tag{1.104}$$

$$G = \tilde{G}, \text{ if } y^* = \infty, \quad \text{and} \quad G(y) = \begin{cases} \tilde{G}(y), & \text{for } y \in [0, y^*], \\ \bar{G}(y), & \text{for } y > y^*, \end{cases} \text{ if } y^* < \infty, \tag{1.105}$$

where $\tilde{G}$ is as in Lemma 6, $\bar{F}$, $\bar{G}$ are as in Lemma 7, and $y^* \equiv \bar{y}^*$, where $\bar{y}^*$ is given by (1.99).

The preceding results determine completely the boundaries of the three possible regions. To specify w inside the "wait" region $\mathcal{W}$, we still have to solve (1.85) and (1.91)-(1.94). To this end, it is straightforward to see that, if the associated integrals are finite, then the function

$$A(y) = \frac{1}{\sigma^2(n-m)} \int_y^\infty \int_{G(u)}^\infty s^{-n-1} \left[H(s,u) - rK^+\right] ds\, du > 0, \quad y \geq 0, \quad (1.106)$$

satisfies (1.85) as well as (1.91) and (1.92). In this expression, the inequality follows thanks to (1.87) or the second inequality in (1.102), depending on the case, and the assumption that $H(\cdot, y)$ is increasing. It is worth noting that adding a constant on the right hand side of (1.106) would yield a further solution to (1.85). However, it turns out that (1.106) gives the *only* solution of (1.85) that renders w compatible with the requirements of the verification theorem that we proved in Section 1.4.

If $y^* < \infty$, then

$$B(y) = -\frac{1}{\sigma^2(n-m)} \int_{y^*}^y \int_0^{F(u)} s^{-m-1} \left[H(s,u) + rK^-\right] ds\, du > 0, \quad y > y^*, \ (1.107)$$

satisfies (1.93) or (1.94). Here, the positivity of B follows from the first inequality in (1.102) and the assumption that $H(\cdot, y)$ is increasing. As above, we have set a possible additive constant to zero because for *no other* choice can the resulting function w be identified with the value function of the control problem.

With reference to (1.79), w must satisfy

$$w(x,y) = w(x, G^{[-1]}(x)) - K^+ \left(G^{[-1]}(x) - y\right), \quad \text{for } (x,y) \in \mathcal{I},$$

where $G^{[-1]}$ is the inverse of the function G. Similarly, if $\mathcal{D} \neq \emptyset$, then (1.89) implies

$$w(x,y) = w(x, \Phi(x)) - K^-(y - \Phi(x)), \quad \text{for } (x,y) \in \mathcal{D},$$

where the function $\Phi :]0, \infty[\to \mathbb{R}_+$ is defined by

$$\Phi(x) = \begin{cases} F^{[-1]}(x), & \text{if } x \geq F(y^*), \\ 0, & \text{if } y^* = 0 \text{ and } F(0) > x. \end{cases} \quad (1.108)$$

Summarising, we have two possibilities. If the point $y^* \equiv \bar{y}^*$ as in (1.72) or (1.99) is equal to ∞, then

$$w(x,y) = \begin{cases} A(y)x^n + R(x,y), & \text{for } (x,y) \text{ such that } 0 < x \le G(y), \\ w(x, G^{[-1]}(x)) - K^+(G^{[-1]}(x) - y), & \text{for } (x,y) \text{ such that } G(y) < x. \end{cases} \tag{1.109}$$

On the other hand, if $y^* < \infty$, then

$$w(x,y) = \begin{cases} w(x, \Phi(x)) - K^-(y - \Phi(x)), & \text{for } (x,y) \text{ s.t. } y > y^*,\ x < F(y), \\ A(y)x^n + R(x,y), & \text{for } (x,y) \text{ s.t. } y \in [0, y^*] \cap \mathbb{R},\ x \le G(y), \\ A(y)x^n + B(y)x^m + R(x,y), & \text{for } (x,y) \text{ s.t. } y > y^*,\ F(y) \le x \le G(y), \\ w(x, G^{[-1]}(x)) - K^+(G^{[-1]}(x) - y), & \text{for } (x,y) \text{ s.t. } G(y) < x. \end{cases} \tag{1.110}$$

It is worth noting that, if $y^* = 0$ and $F(0) > 0$, then (1.76) and (1.107) imply

$$w(x,0) = A(0)x^n + R(x,0), \quad \text{for } 0 < x \le G(0),$$

which is consistent with (1.110).

The following result is concerned with proving that the construction above indeed provides a solution to the HJB equation (1.47)–(1.48), as well as with certain estimates that we shall need.

Lemma 8 *Suppose that Assumptions 1 and 2 hold. The function w given by (1.109)–(1.110), where F, G and A, B are as in (1.104), (1.105) and (1.106), (1.107), respectively, is C^2 and satisfies the HJB equation (1.47)–(1.48). Also, w satisfies*

$$w(x,y) \le C_5 \left(1 + y + G^{n-\varepsilon_4}(y) + G^{\alpha}(y)y^{\beta} + x^{n-\varepsilon_4}\right), \quad \textit{for all } (x,y) \in \mathcal{S}, \tag{1.111}$$

for some constants $C_5 > 0$ and $\varepsilon_4 \in]0, n[$, as well as (1.52) in the verification Theorem 5.

Remark 2 A careful inspection of the proof of this result reveals that, had we perturbed the expressions on the right hand sides of (1.106) and (1.107) by additive constants, we would still have obtained a further solution to the HJB equation (1.47)–(1.48). However, such a solution would not satisfy an estimate such as the one provided by (1.111) that plays a fundamental role in the proof of the verification Theorem 5.

Theorem 9 *Consider the capacity control problem formulated in Section 1.2, and suppose that Assumptions 1 and 2 hold. The value function v identifies with the function w given by (1.109)-(1.110), where F, G and A, B are as in (1.104), (1.105) and (1.106), (1.107), respectively. The optimal capacity process Y° reflects the joint process (X, Y°) along the boundaries G and F in the positive and in the negative y-direction, respectively, and can be constructed as in the proof below.*

Proof. With regard to Lemma 8, we only have to construct a process Y° such that (1.53)–(1.57) in the verification Theorem 5 hold. To this end, we construct Y° so that the process (X, Y°) is reflecting along the boundaries G and F in the positive and in the negative y-direction, respectively, as follows. First, we define

$$\tau_0 = \inf\{t \geq 0 :\ X_t \geq G(y)\} \quad \text{and} \quad Y_t^{(1)} = y\mathbf{1}_{\{t\leq\tau_0\}} + G^{[-1]}\left(\sup_{s\leq t} X_s\right)\mathbf{1}_{\{\tau_0<t\}},$$

where $G^{[-1]}$ is the inverse function of G. If $y^* = \infty$, then $Y^{\circ} = Y^{(1)}$. If $y^* < \infty$, then we define inductively the $(\mathcal{F}_t)$-stopping times τ_n and the processes $Y^{(n)}$ by

$$\begin{aligned}
\tau_{2k+1} &= \inf\left\{t \geq 0 :\ X_t < \hat{F}\left(Y_t^{(2k+1)}\right)\right\}, \\
Y_t^{(2k+2)} &= Y_t^{(2k+1)}\mathbf{1}_{\{t\leq\tau_{2k+1}\}} + \Phi\left(\inf_{\tau_{2k+1}<s\leq t} X_s\right)\mathbf{1}_{\{\tau_{2k+1}<t\}},
\end{aligned}$$

for $k = 0, 1, \ldots$, where

$$\hat{F}(y) = \begin{cases} 0, & \text{if } y < y^*, \\ F(y), & \text{if } y \geq y^*, \end{cases}$$

and Φ is defined by (1.108), and by

$$\begin{aligned}
\tau_{2k} &= \inf\left\{t \geq 0 :\ X_t > G\left(Y_t^{(2k)}\right)\right\}, \\
Y_t^{(2k+1)} &= Y_t^{(2k)}\mathbf{1}_{\{t\leq\tau_{2k}\}} + G^{[-1]}\left(\sup_{\tau_{2k}<s\leq t} X_s\right)\mathbf{1}_{\{\tau_{2k}<t\}},
\end{aligned}$$

for $k = 1, 2, \ldots$. Observing that $\lim_{n\to\infty}\tau_n = \infty$, P-a.s., and that $Y_t^{(n)} = Y_t^{(n+1)}$, for all $t \in [0, \tau_{n+1}]$, we define Y° by $Y_t^{\circ} = Y_t^{(n)}$ for $t < \tau_n$.

In either case, we can see that (1.53) is satisfied, and, if $\xi^{\circ+}$ and $\xi^{\circ-}$ are the increasing processes providing the minimal decomposition of Y° into $Y^{\circ} = y + \xi^{\circ+} - \xi^{\circ-}$, then both of (1.54) and (1.55) hold. Also, in either case, we can see that

$$Y_t^{\circ} \leq y\mathbf{1}_{\{\bar{X}_t \leq G(y)\}} + G^{[-1]}(\bar{X}_t)\mathbf{1}_{\{\bar{X}_t > G(y)\}}, \tag{1.112}$$

where $\bar{X}_t = \sup_{s \leq t} X_s$. Combining this inequality with the definition (1.105) of G and the estimates in (1.88) and (1.103), we can see that

$$Y_t^{\circ} \leq y\mathbf{1}_{\{\bar{X}_t \leq G(y)\}} + C_4 \bar{X}_t^{\alpha/(1-\beta)}\mathbf{1}_{\{\bar{X}_t > G(y)\}} \tag{1.113}$$

and

$$\xi_t^{\circ +} \leq C_4 \bar{X}_t^{\alpha/(1-\beta)}. \tag{1.114}$$

Now, we can use (1.113), the observation that

$$G(Y_t^{\circ}) \leq G(y)\mathbf{1}_{\{\bar{X}_t \leq G(y)\}} + \bar{X}_t \mathbf{1}_{\{\bar{X}_t > G(y)\}},$$

which follows immediately from (1.112), to see that, e.g.,

$$\begin{aligned} G^{\alpha}\left(Y_t^{\circ}\right)\left(Y_t^{\circ}\right)^{\beta} &\leq G^{\alpha}(y) y^{\beta}\mathbf{1}_{\{\bar{X}_t \leq G(y)\}} + C_4^{\beta} \bar{X}_t^{\alpha/(1-\beta)}\mathbf{1}_{\{\bar{X}_t > G(y)\}} \\ &\leq G^{\alpha}(y) y^{\beta} + C_4^{\beta} \bar{X}_t^{\alpha/(1-\beta)}. \end{aligned}$$

In view of this and similar calculations involving the other terms, as well as the estimate (1.111) and the fact that $\alpha < \frac{\alpha}{1-\beta} < n$ (see Assumption 2), we can conclude that (1.113)–(1.114) imply that the estimates (1.56)–(1.57) hold true, and the proof is complete. □

We conclude our analysis above with the following result.

Corollary 10 *Suppose that h is given by (1.27) in Example 1, and $K^+, K^+ + K^- > 0$. We have the following two cases:*

(a) If $K^- \geq 0$, then $y^* = \infty$,

$$G(y) = \left[-\frac{rK^+(\alpha - m)}{m\beta}\right]^{1/\alpha} y^{(1-\beta)/\alpha}, \tag{1.115}$$

and the optimal strategy can be depicted by Figure 1.1.

(b) If $K^- < 0$, then $y^* = 0$ and

$$\lim_{y \downarrow 0} F(y) = \lim_{y \downarrow 0} G(y) = 0. \tag{1.116}$$

and the optimal strategy can be depicted by Figure 1.4. In either case, $v < \infty$ if $\frac{\alpha}{1-\beta} < n$, and $v \equiv \infty$ if $\frac{\alpha}{1-\beta} > n > \alpha$, where n is the positive solution of (1.13).

Proof. As we have observed in Example 1, Assumptions 1 and 2 are satisfied and $v < \infty$ if and only if $\frac{\alpha}{1-\beta} < n$. Also, if $\frac{\alpha}{1-\beta} > n > \alpha$, then we have proved in Lemma 3 that $v \equiv \infty$.

The condition distinguishing the two cases follows from a simple inspection of (1.99), while showing (1.115) involves elementary calculations. To see (1.116), we observe that the system of equations (1.97)–(1.98), which specifies F and G, is equivalent to

$$\frac{\beta}{\alpha - m} y^{-(1-\beta)} \left[G^{\alpha-m}(y) - F^{\alpha-m}(y)\right] = -\frac{r}{m} \left[K^+ G^{-m}(y) + K^- F^{-m}(y)\right], \tag{1.117}$$

$$\frac{\beta}{n - \alpha} y^{-(1-\beta)} \left[G^{\alpha-n}(y) - F^{\alpha-n}(y)\right] = \frac{r}{n} \left[K^+ G^{-n}(y) + K^- F^{-n}(y)\right]. \tag{1.118}$$

Since $m < 0 < \alpha, 1-\beta$ and F, G are increasing, the right hand side of (1.117) remains bounded as $y \downarrow 0$, and $\lim_{y \downarrow 0} y^{-(1-\beta)} = \infty$. It follows that (1.117) cannot be true unless (1.116) is satisfied, and the proof is complete. □

Remark 3 In the context of the special case considered in Corollary 10, it is worth noting that the solution w to the HJB equation (1.47)–(1.48) that we have constructed following intuition based on economical considerations is finite for all $\alpha > 0$ and $\beta \in]0,1[$. Had we adopted a *formal* approach, this observation would have suggested the adoption of the capacity expansion strategy that keeps the process (X, Y) inside the "wait" region $\mathcal{W}$ that is determined by the functions F and G provided by the unique solution to the associated free-boundary problem. However, such a formal approach would have lead us to wrong conclusions because

$$w(x,y) < \infty = v(x,y), \quad \text{for all } (x,y) \in \mathcal{S},$$

if $\frac{\alpha}{1-\beta} > n$.

Remark 4 In the special case of Corollary 10 arising when $\alpha = 1 - \beta$ and $K^- < 0$, we can verify that (1.117) and (1.118) are satisfied by the functions

$$F(y) = \kappa y \quad \text{and} \quad G(y) = \nu y, \quad \text{for } y \geq 0,$$

where κ and ν are constants satisfying the system of algebraic equations

$$\frac{1-\alpha}{\alpha - m}\left[\nu^{\alpha-m} - \kappa^{\alpha-m}\right] = -\frac{r}{m}\left[K^+\nu^{-m} + K^-\kappa^{-m}\right], \tag{1.119}$$

$$\frac{1-\alpha}{n-\alpha}\left[\nu^{-(n-\alpha)} - \kappa^{-(n-\alpha)}\right] = \frac{r}{n}\left[K^+\nu^{-n} + K^-\kappa^{-n}\right]. \tag{1.120}$$

Abel and Eberly [AE96] have considered this special case with $r > b$, which satisfies our assumptions thanks to the equivalence $r > b \Leftrightarrow n > 1$, and have proved that the system of equations (1.119)–(1.120) has a unique solution such that $0 < \kappa < \nu$.

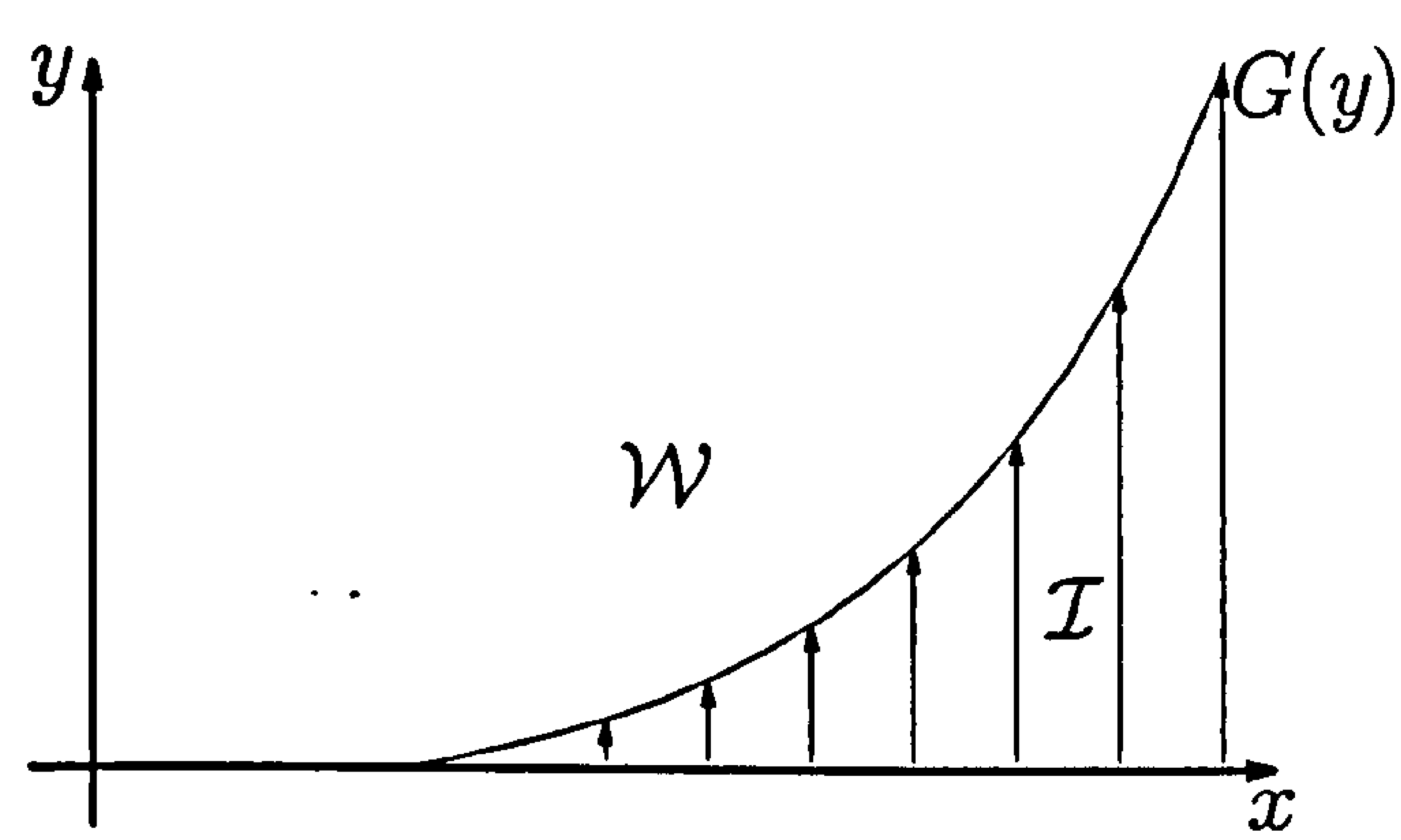

Figure 1.1: A possible optimal capacity control strategy. In this case, it is never optimal to decrease the project's capacity.

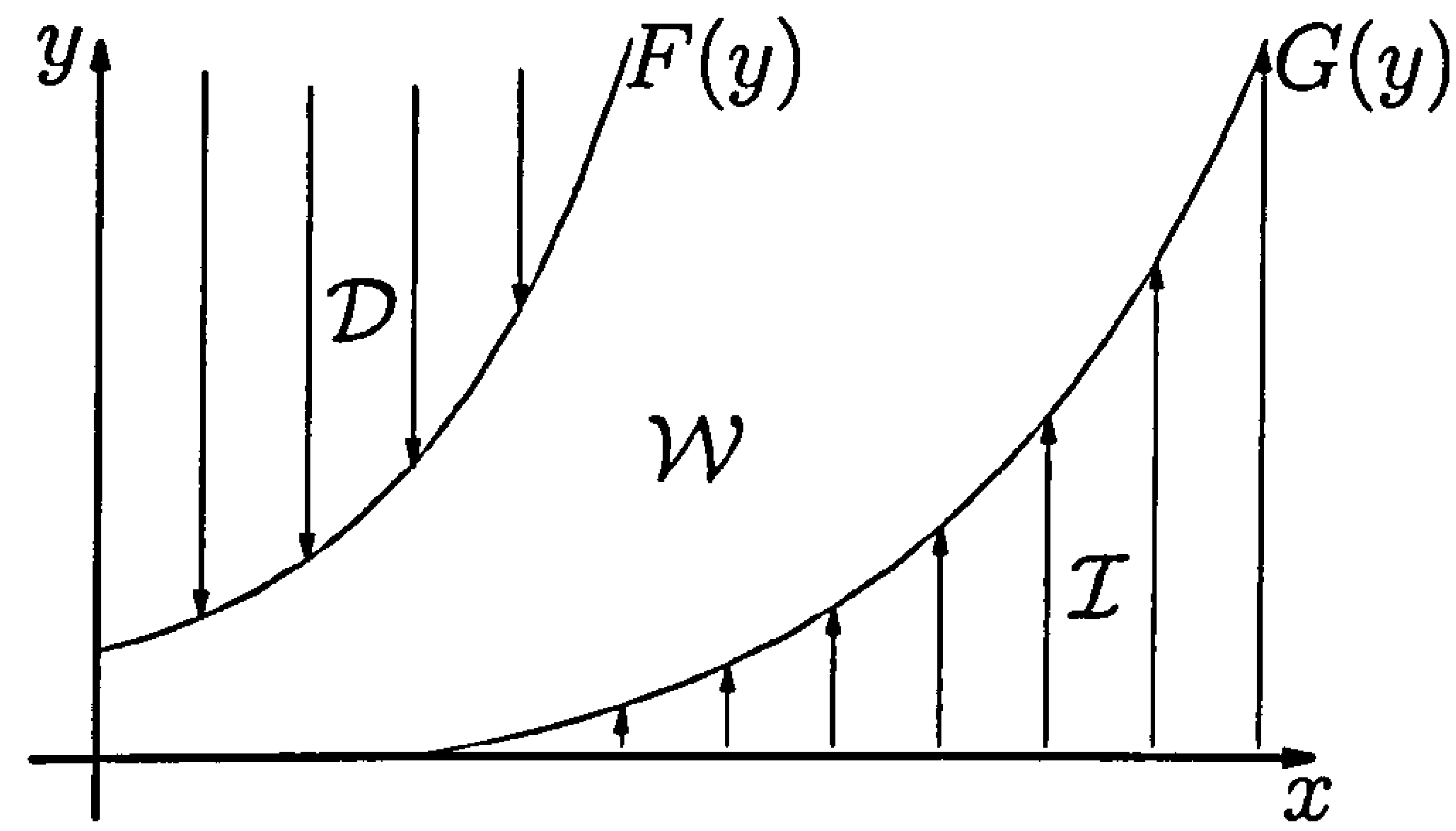

Figure 1.2: A possible optimal capacity control strategy. In this case, increasing the project's capacity, waiting and decreasing the project's capacity are all parts of the optimal strategy. Also, the point y^* defined by (1.72) is strictly positive, and $F(0) = 0$.

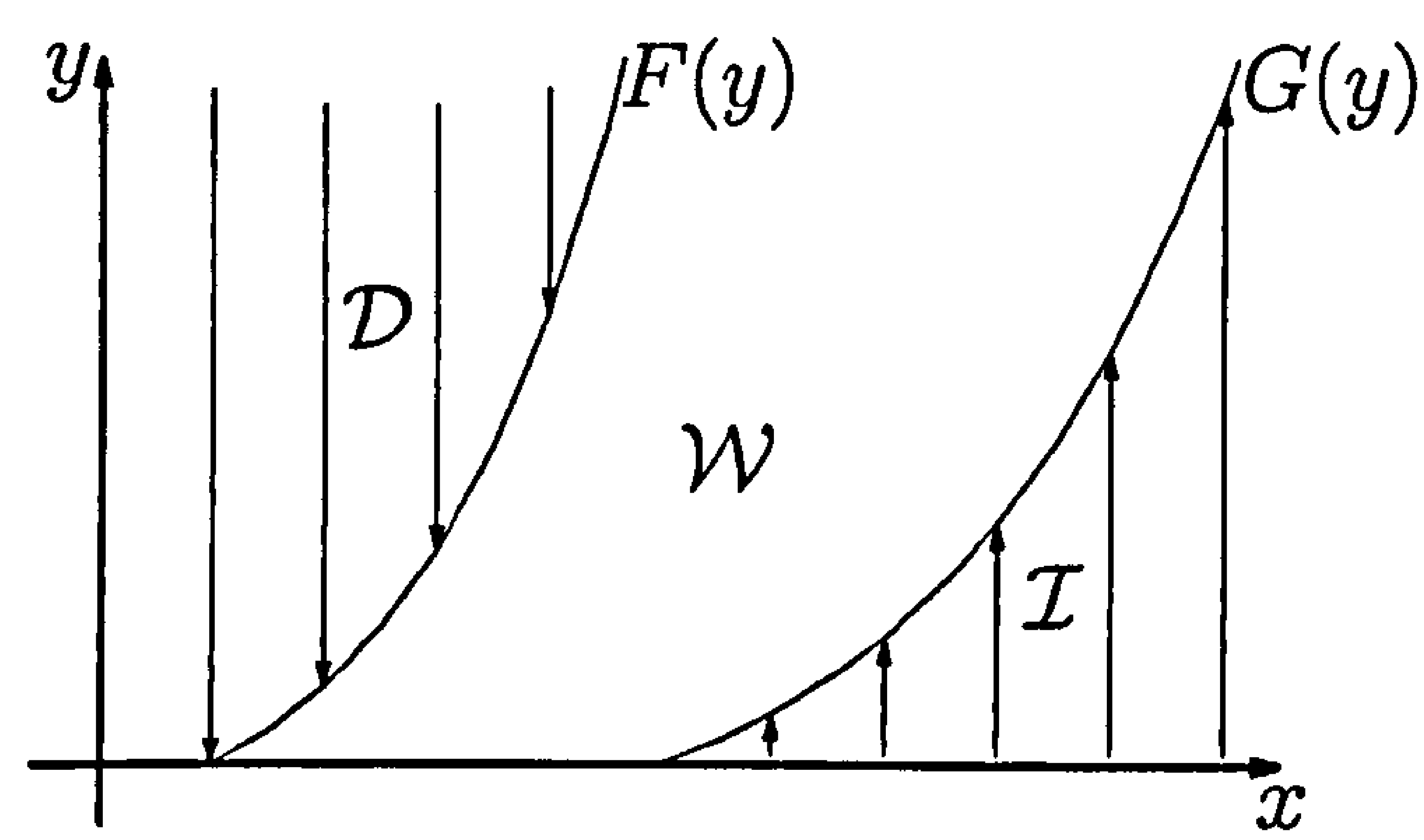

Figure 1.3: A possible optimal capacity control strategy. In this case, increasing the project's capacity, waiting and decreasing the project's capacity all belong to the set of optimal tactics. Also, $y^* = 0$, where y^* is defined by (1.72), $F(0) > 0$, and $\{(x, 0) : x \leq F(0)\}$ is a subset of the waiting region $\mathcal{W}$.

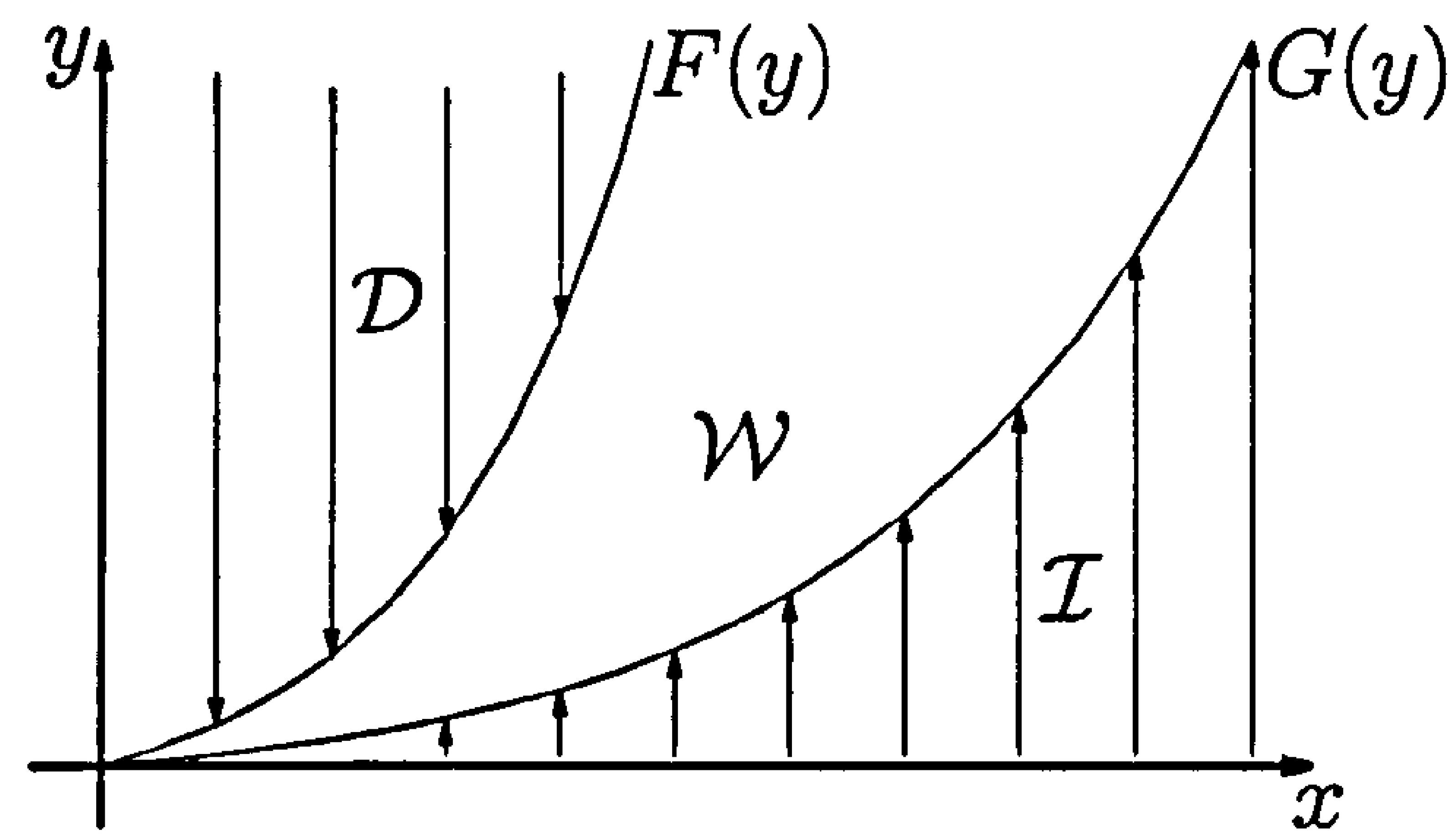

Figure 1.4: A possible optimal capacity control strategy. This case arises when the running payoff function h identifies with the Cobb-Douglas production function and $K^- < 0$.

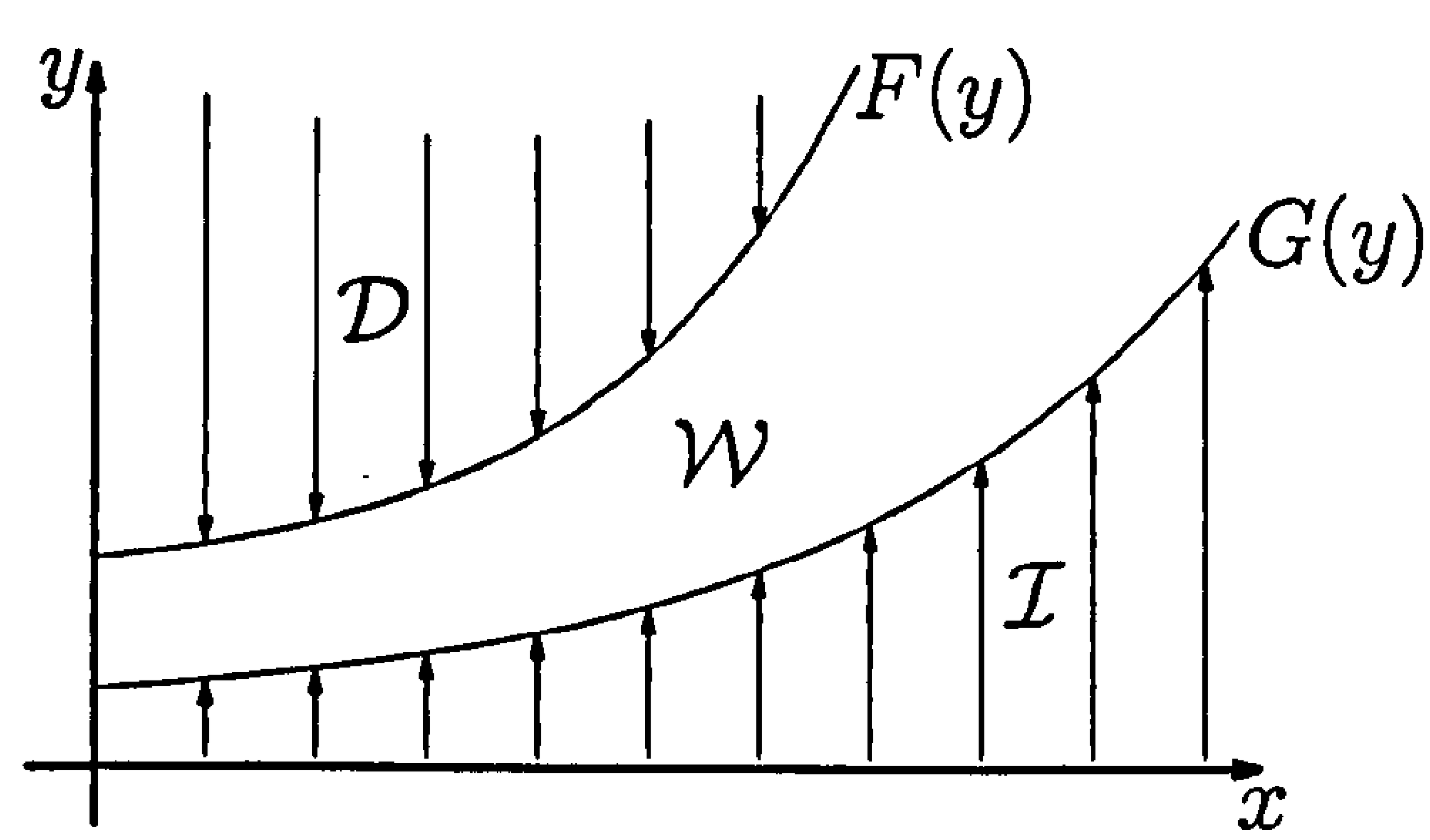

Figure 1.5: A possible optimal capacity control strategy. This case *cannot* arise under our assumptions.

Appendix: Proof of selected results

Proof of Lemma 6. Suppose that (1.21) in Assumption 1 is satisfied. Fix any $y \geq 0$, and suppose that $\inf_{x>0} H(x,y) - rK^+ \geq 0$. In this case, $H(x,y) - rK^+ > 0$, for all $x > 0$, because $H(\cdot, y)$ is a strictly increasing function. This implies that $q(x,y) > 0$, for all $x > 0$, and, therefore, the equation $q(x,y) = 0$ has no solution $x > 0$.

Now, fix any $y \geq 0$, and assume that $\inf_{x>0} H(x,y) < rK^+$. Recalling the assumption that $H(\cdot, y)$ is strictly increasing, we define

$$x^\dagger = x^\dagger(y) := \inf\left\{x > 0 : \; H(x,y) - rK^+ > 0\right\} > 0,$$

and we observe that

$$\frac{\partial}{\partial x} q(x,y) = x^{-m-1}\left[H(x,y) - rK^+\right] \begin{cases} < 0, & \text{for all } x \in]0, x^\dagger[, \\ > 0, & \text{for all } x > x^\dagger. \end{cases} \tag{1.121}$$

Combining the fact that $q(\cdot, y)$ is strictly decreasing in $]0, x^\dagger[$ and strictly increasing in $]x^\dagger, \infty[$, with $q(0, y) = 0$, we can see that $q(x, y) < 0$, for all $x \leq x^\dagger$. In particular, $q(x^\dagger, y) < 0$. Therefore, if $q(x, y) = 0$ has a solution $x > 0$ then this must satisfy $x > x^\dagger$. Also, given that it exists, this solution is unique because $q(\cdot, y)$ is strictly increasing in $]x^\dagger, \infty[$. To prove that the required solution indeed exists, it suffices to show that $\lim_{x\to\infty} q(x, y) = \infty$. The assumption that $\lim_{x\to\infty} H(x, y) = \infty$ implies that, given any constant $M > 0$, there exists $\gamma > x^\dagger$ such that $H(x, y) - rK^+ \geq M$, for all $x \geq \gamma$. However, given any such choice of these constants, we calculate

$$\begin{aligned} \lim_{x\to\infty} q(x,y) &= \lim_{x\to\infty}\left[q(\gamma, y) + \int_\gamma^x s^{-m-1}\left[H(s,y) - rK^+\right] ds\right] \\ &\geq \lim_{x\to\infty}\left[q(\gamma, y) + \frac{M}{m}\gamma^{-m} - \frac{M}{m}x^{-m}\right] = \infty. \end{aligned}$$

If (1.22) in Assumption 1 also holds and the point $\tilde{y}_*$ defined as in (1.86) is finite, then $\inf_{x>0} H(x, y) < rK^+$, for all $y > \tilde{y}_*$. It follows that equation (1.83) defines uniquely a continuous function $\tilde{G} :]\tilde{y}_*, \infty[\to]0, \infty[$. Moreover, the arguments above regarding the solvability of $q(x, y) = 0$ imply (1.87).

To see that $\tilde{G}$ is C^1 and strictly increasing, we differentiate $q(\tilde{G}(y), y) = 0$ with respect to y to obtain

$$\tilde{G}'(y) = -\tilde{G}^{m+1}(y)\left[H(\tilde{G}(y), y) - rK^+\right]^{-1} \int_0^{\tilde{G}(y)} s^{-m-1} H_y(s, y)\, ds > 0, \tag{1.122}$$

for all $y > \tilde{y}_*$. The inequality here follows thanks to (1.87) and (1.22) in Assumption 1.

Now, suppose that (1.25) in Assumption 2 also holds and observe that this implies

$$\inf_{x>0} H(x,y) < rK^+, \quad \text{for all } y > 0.$$

However, this inequality implies that $\tilde{y}_* = 0$. Finally, with regard to (1.25) in Assumption 2 and (1.121) above, we calculate

$$\frac{\partial}{\partial x} q(x,y) \le x^{-m-1} \left[\beta C x^\alpha y^{-(1-\beta)} - r\vartheta\right].$$

Combining this inequality with $q(0,y) = 0$, we can see that, given any $y > 0$, $\tilde{G}(y)$ is greater than or equal to the strictly positive solution of the equation

$$\int_0^z s^{-m-1} \left[\beta C s^\alpha y^{-(1-\beta)} - r\vartheta\right] ds = 0,$$

which yields

$$\tilde{G}(y) \ge \left(-\frac{r\vartheta(\alpha - m)}{\beta C m}\right)^{\frac{1}{\alpha}} y^{\frac{1-\beta}{\alpha}}, \quad \text{for all } y > 0.$$

However, this implies (1.88). □

Proof of lemma 7. Suppose that Assumption 1 holds. To study the solvability of the system of equations (1.95) and (1.96), we first prove that (1.95) defines uniquely a mapping $L : (\mathbb{R} \setminus \{0\})^2 \to]0,\infty[$ such that

$$f(x_1, L(x_1,y), y) = 0 \quad \text{and} \quad L(x_1,y) > x_1. \tag{1.123}$$

To this end, fix any $x_1 > 0$, $y > 0$, and observe that

$$f(x_1, x_1, y) = -\frac{1}{m} r \left(K^+ + K^-\right) x_1^{-m} > 0. \tag{1.124}$$

Given $M > 0$, observe that the assumption that $\lim_{x\to\infty} H(x,y) = \infty$, for all $y > 0$, implies that there exists a constant $\gamma > x_1$ such that $H(x,y) - rK^+ \ge M$, for all $x \ge \gamma$. For such a choice of parameters, since $m < 0$, we calculate

$$\begin{aligned}
\lim_{x_2\to\infty} f(x_1,x_2,y) &= \lim_{x_2\to\infty} \Bigg[-\int_{x_1}^{\gamma} s^{-m-1} \left[H(s,y) - rK^+\right] ds \\
&\qquad - \int_{\gamma}^{x_2} s^{-m-1} \left[H(s,y) - rK^+\right] ds - \frac{r}{m}\left(K^+ + K^-\right) x_1^{-m} \Bigg] \\
&\le \lim_{x_2\to\infty} \left[f(x_1,\gamma,y) - M \int_{\gamma}^{x_2} s^{-m-1}\, ds \right] \\
&= \lim_{x_2\to\infty} \left[f(x_1,\gamma,y) - \frac{M}{m}\gamma^{-m} + \frac{M}{m} x_2^{-m} \right] \\
&= -\infty.
\end{aligned} \tag{1.125}$$

Also, it is straightforward to calculate

$$\frac{\partial f}{\partial x_2}(x_1, x_2, y) = -x_2^{-m-1}\left[H(x_2, y) - rK^+\right] \begin{cases} > 0, & \text{for all } x_2 \in]0, x^\dagger[, \\ < 0, & \text{for all } x_2 > x^\dagger, \end{cases} \tag{1.126}$$

where

$$x^\dagger = x^\dagger(y) := \inf\left\{x > 0 : \; H(x, y) - rK^+ > 0\right\}.$$

Combining the fact that $f(x_1, \cdot, y)$ is strictly increasing in the interval $[x_1, x^\dagger[$, if $x_1 < x^\dagger$, and strictly decreasing in the interval $]x^\dagger \vee x_1, \infty[$, with (1.125) and (1.124), we can conclude that the equation $f(x_1, x_2, y) = 0$ has a unique solution $x_2 = L(x_1, y)$ which satisfies (1.123) as well as

$$H(L(x_1, y), y) - rK^+ > 0. \tag{1.127}$$

Furthermore, differentiation of $f(x_1, L(x_1, y), y) = 0$ with respect to x_1 yields

$$\frac{\partial}{\partial x_1} L(x_1, y) = \frac{x_1^{-m-1}\left[H(x_1, y) + rK^-\right]}{L^{-m-1}(x_1, y)\left[H(L(x_1, y), y) - rK^+\right]}, \tag{1.128}$$

while differentiation of $f(x_1, L(x_1, y), y) = 0$ with respect to y gives

$$\frac{\partial}{\partial y} L(x_1, y) = -L^{m+1}(x_1, y)\left[H(L(x_1, y), y) - rK^+\right]^{-1} \int_{x_1}^{L(x_1, y)} s^{-m-1} H_y(s, y)\, ds. \tag{1.129}$$

To prove that the system of equations (1.95) and (1.96) has a unique solution (x_1, x_2) such that $0 < x_1 < x_2$ we have to show that there exists a unique $x_1 > 0$ such that $g(x_1, L(x_1, y), y) = 0$. To this end, we first observe that the calculation

$$g(x_1, L(x_1, y), y) = \int_{x_1}^{L(x_1, y)} s^{-n-1}\left[H(s, y) - rK^+\right] + \frac{1}{n} r\left(K^+ + K^-\right) x_1^{-n}$$

and the assumptions $\lim_{x \to \infty} H(x, y) = \infty$, $K^+ + K^- > 0$ imply that

there exists a constant $N > 0$ such that $g(x_1, L(x_1, y), y) > 0$, for all $x_1 \geq N$(1.130)

Now, with regard to (1.128), we calculate

$$\frac{\partial}{\partial x_1} g(x_1, L(x_1, y), y) = x_1^{-m-1}\left[L^{m-n}(x_1, y) - x_1^{m-n}\right]\left[H(x_1, y) + rK^-\right]. \tag{1.131}$$

Since $L(x_1, y) > x_1$ and $m < n$, $L^{m-n}(x_1, y) - x_1^{m-n} < 0$. Therefore, if $\inf_{x>0} H(x, y) \geq -rK^-$, then $g(\cdot, L(\cdot, y), y)$ is decreasing, which, combined with (1.130), implies that the equation $g(x_1, L(x_1, y), y) = 0$ cannot have a solution $x_1 > 0$. Therefore, we must have $\inf_{x>0} H(x, y) < -rK^-$. Assuming that this condition holds, we recall that $H(\cdot, y)$ is strictly increasing, we define

$$x^\ddagger = x^\ddagger(y) := \inf \left\{ x > 0 : \ H(x, y) + rK^- > 0 \right\},$$

and we observe that

$$g(\cdot, L(\cdot, y), y) \text{ is strictly increasing in }]0, x^\ddagger[\text{ and strictly decreasing in }]x^\ddagger, \infty[\tag{1.132}$$

Furthermore, under this condition, there exist $\varepsilon > 0$ and $\delta < x^\ddagger$ such that $H(x_1, y) + rK^- \leq -\varepsilon$, for all $x_1 \leq \delta$. For such a choice of parameters, we calculate

$$\begin{aligned}
\lim_{x_1 \downarrow 0} & \int_{x_1}^{\infty} s^{-n-1} \left[H(s, y) + rK^- \right] ds \\
& \leq \lim_{x_1 \downarrow 0} \left[\frac{\varepsilon}{n} \delta^{-n} - \frac{\varepsilon}{n} x_1^{-n} + \int_{\delta}^{\infty} s^{-n-1} \left[H(s, y) + rK^- \right] ds \right] \\
& = -\infty.
\end{aligned} \tag{1.133}$$

In view of this, (1.127), and the assumption that $H(\cdot, y)$ is increasing,

$$\begin{aligned}
\lim_{x_1 \downarrow 0} & g(x_1, L(x_1, y), y) \\
& = \lim_{x_1 \downarrow 0} \left[\int_{x_1}^{\infty} s^{-n-1} \left[H(s, y) + rK^- \right] ds - \int_{L(x_1, y)}^{\infty} s^{-n-1} \left[H(s, y) - rK^+ \right] ds \right] \\
& \leq \lim_{x_1 \downarrow 0} \int_{x_1}^{\infty} s^{-n-1} \left[H(s, y) + rK^- \right] ds \\
& = -\infty.
\end{aligned} \tag{1.134}$$

However, combining (1.130), (1.132) and (1.134), we can see that the equation $g(x_1, L(x_1, y), y) = 0$ has a unique solution $x_1 > 0$, which also satisfies

$$H(x_1, y) + rK^- < 0. \tag{1.135}$$

Summarising the analysis above, under the assumption that the point $\bar{y}^*$ defined as in (1.99) is finite, the system of equations (1.95) and (1.96) defines uniquely two continuous functions $\bar{F}, \bar{G} :]\bar{y}^*, \infty[\to]0, \infty[$ that satisfy $\bar{F}(y) < \bar{G}(y)$, for all $y > \bar{y}^*$, as well as (1.102). Also, (1.100)–(1.101) follow from a simple continuity argument combining the definition of $\bar{y}^*$ and (1.135).

Now, assuming that $\bar{y}^* < \infty$, consider any point $y > \bar{y}^*$. Differentiating the equation $g(\bar{F}(y), L(\bar{F}(y), y), y) = 0$ with respect to y, using (1.128), and observing that $\bar{G}(y) = L(\bar{F}(y), y)$, we calculate

$$\begin{aligned} \bar{F}'(y) = & -\bar{F}^{m+1}(y)\bar{G}^{-n}\left[\bar{G}^{-(n-m)}(y) - \bar{F}^{-(n-m)}(y)\right]^{-1}\left[H(\bar{F}(y), y) + rK^-\right]^{-1} \\ & \times \int_{\bar{F}(y)}^{\bar{G}(y)} \left[\left(\frac{\bar{G}(y)}{s}\right)^n - \left(\frac{\bar{G}(y)}{s}\right)^m\right] \frac{1}{s} H_y(s, y)\, ds > 0, \end{aligned} \tag{1.136}$$

the inequality following thanks to assumption (1.22), the first inequality in (1.102) and the fact that $m < 0 < n$. Also, differentiating the equation $f(\bar{F}(y), L(\bar{F}(y), y), y) = 0$ with respect to y, and using (1.129) and (1.136), we calculate

$$\begin{aligned} \bar{G}'(y) = & -\bar{F}^{-n}(y)\bar{G}^{m+1}\left[\bar{G}^{-(n-m)}(y) - \bar{F}^{-(n-m)}(y)\right]^{-1}\left[H(\bar{G}(y), y) - rK^+\right]^{-1} \\ & \times \int_{\bar{F}(y)}^{\bar{G}(y)} \left[\left(\frac{\bar{F}(y)}{s}\right)^n - \left(\frac{\bar{F}(y)}{s}\right)^m\right] \frac{1}{s} H_y(s, y)\, ds > 0, \end{aligned}$$

the inequality following thanks to (1.102) and (1.22).

Finally, suppose that (1.25) in Assumption 2 is also true. With reference to the equation $f(\bar{F}(y), \bar{G}(y), y) = 0$, we calculate

$$\begin{aligned} 0 = & -\int_{\bar{F}(y)}^{\bar{G}(y)} s^{-m-1}\left[H(s, y) - rK^+\right] ds - \frac{1}{m} r\left(K^+ + K^-\right)\bar{F}^{-m}(y) \\ \geq & -\left[\frac{\beta C}{\alpha - m}\bar{G}^{\alpha-m}(y) y^{-(1-\beta)} + \frac{r\vartheta}{m}\bar{G}^{-m}(y)\right] \\ & + \left[\frac{\beta C}{\alpha - m}\bar{F}^{\alpha-m}(y) y^{-(1-\beta)} - \frac{1}{m} r\left(K^+ + K^- - \vartheta\right)\bar{F}^{-m}(y)\right]. \end{aligned}$$

Since $\vartheta < K^+ + K^-$ by assumption, the second term on the right hand side of this expression is strictly positive. Therefore, we must have

$$\frac{\beta C}{\alpha - m}\bar{G}^{\alpha-m}(y) y^{-(1-\beta)} + \frac{r\vartheta}{m}\bar{G}^{-m}(y) > 0.$$

This inequality can be true only if $\bar{G}(y)$ is strictly greater than the unique strictly positive solution of the equation

$$\frac{\beta C}{\alpha - m} z^{\alpha-m} y^{-(1-\beta)} + \frac{r\vartheta}{m} z^{-m} = 0,$$

which yields

$$\bar{G}(y) \geq \left(-\frac{r\vartheta(\alpha - m)}{\beta C m}\right)^{\frac{1}{\alpha}} y^{\frac{1-\beta}{\alpha}}, \quad \text{for all } y > \bar{y}^*.$$

However, this implies (1.103). □

Proof of lemma 8. Consider (1.106), and note that the upper bound in (1.25) in Assumption 2 implies

$$0 < A(y) \leq \frac{\beta C}{\sigma^2(n-m)(n-\alpha)} \int_y^\infty u^{-(1-\beta)} G^{-(n-\alpha)}(u)\, du. \tag{1.137}$$

Recalling the inequalities $\alpha < \frac{\alpha}{1-\beta} < n$, we fix any $\varepsilon_0 > 0$ such that

$$\varepsilon_0 < n - \frac{\alpha}{1-\beta} < n - \alpha.$$

Using the fact that G is increasing and the estimate provided by (1.88) and (1.103), we calculate

$$\begin{aligned}\int_y^\infty u^{-(1-\beta)} G^{-(n-\alpha)}(u)\, du &\leq G^{-\varepsilon_0}(y) \int_y^\infty u^{-(1-\beta)} G^{-(n-\alpha-\varepsilon_0)}(u)\, du \\ &\leq \frac{\alpha C_4^{(1-\beta)(n-\alpha-\varepsilon_0)/\alpha}}{(1-\beta)(n-\varepsilon_0)-\alpha} G^{-\varepsilon_0}(y) y^{-[(1-\beta)(n-\varepsilon_0)-\alpha]/\alpha},\end{aligned}$$

which implies

$$\int_y^\infty u^{-(1-\beta)} G^{-(n-\alpha)}(u)\, du \leq \frac{\alpha C_4^{(1-\beta)(n-\alpha-\varepsilon_0)/\alpha}}{(1-\beta)(n-\varepsilon_0)-\alpha} G^{-\varepsilon_0}(y), \quad \text{for all } y \geq 1. \tag{1.138}$$

Also, the fact that G is increasing implies that

$$\begin{aligned}G^n(y) \int_y^1 u^{-(1-\beta)} G^{-(n-\alpha)}(u)\, du &\leq G^\alpha(y) \int_y^1 u^{-(1-\beta)}\, du \\ &\leq \frac{1}{\beta} G^\alpha(1), \quad \text{for all } y < 1.\end{aligned} \tag{1.139}$$

However, (1.137)-(1.139) imply

$$\begin{aligned}A(y)x^n &\leq A(y)G^n(y) \\ &\leq \frac{\beta C}{\sigma^2(n-m)(n-\alpha)} \left[\frac{\alpha C_4^{(1-\beta)(n-\alpha-\varepsilon_0)/\alpha}}{(1-\beta)(n-\varepsilon_0)-\alpha} G^{n-\varepsilon_0}(y) \mathbf{1}_{\{y\geq 1\}}\right. \\ &\quad \left. + \left(\frac{\alpha C_4^{(1-\beta)(n-\alpha-\varepsilon_0)/\alpha}}{(1-\beta)(n-\varepsilon_0)-\alpha} G^{n-\varepsilon_0}(1) + \frac{1}{\beta} G^\alpha(1)\right) \mathbf{1}_{\{y<1\}}\right] \\ &= C_{51}\left(1 + G^{n-\varepsilon_0}(y)\right), \quad \text{for all } y \geq 0 \text{ and } x \leq G(y),\end{aligned} \tag{1.140}$$

where $C_{51} > 0$ is a constant.

If $y^* < \infty$, then (1.107), the assumption that $K^+ + K^- > 0$, the lower bound in (1.25) in Assumption 2 and the fact that F is increasing imply that, given any $y > y^*$,

$$B(y) \leq -\frac{C + rK^+}{\sigma^2 m(n-m)} \int_{y^*}^{y} F^{-m}(u)\, du$$
$$\leq -\frac{C + rK^+}{\sigma^2 m(n-m)} y F^{-m}(y).$$

In light of this calculation and the fact that $m < 0$, we can see that

$$\sup_{x \in [F(y), G(y)]} B(y) x^m \leq B(y) F^m(y) \leq C_{52} y, \quad \text{for all } y > y^*, \tag{1.141}$$

where $C_{52} > 0$ is a constant. Since R is increasing in x (see (1.26) in Assumption 2 and (1.17)), the upper bound in Lemma 2 implies

$$\sup_{x \leq G(y)} R(x,y) \leq R(G(y), y)$$
$$\leq C_1 \left(1 + y + G^{n-\vartheta}(y) + G^{\alpha}(y) y^{\beta}\right), \quad \text{for all } y \geq 0.$$

However, combining this estimate with (1.140) and (1.141), we can see that w satisfies

$$w(x,y) \leq C_{53} \left(1 + y + G^{n-\varepsilon_0 \wedge \vartheta}(y) + G^{\alpha}(y) y^{\beta}\right), \quad \text{for all } (x,y) \in \overline{\mathcal{W}}, \tag{1.142}$$

for some constant $C_{53} > 0$. With regard to the structure of w provided by (1.109)–(1.110), this inequality and the estimates provided by (1.88) and (1.103) imply

$$\begin{aligned} w(x,y) &\leq w(x, G^{[-1]}(x)) + K^+ y \\ &\leq C_{53} \left(1 + G^{[-1]}(x) + x^{n-\varepsilon_0 \wedge \vartheta} + x^{\alpha} \left[G^{[-1]}(x)\right]^{\beta}\right) + K^+ y \\ &\leq C_{54} \left(1 + y + x^{n-\varepsilon_0 \wedge \vartheta} + x^{\alpha/(1-\beta)}\right), \quad \text{for } (x,y) \in \mathcal{I}, \end{aligned} \tag{1.143}$$

for some constant $C_{54} > 0$. Also, since $\Phi(x) \leq y$, for all $(x,y) \in \mathcal{D}$, and G is increasing,

$$\begin{aligned} w(x,y) &\leq w(x, \Phi(x)) + |K^-| y \\ &\leq C_{53} \left(1 + \Phi(x) + G^{n-\varepsilon_0 \wedge \vartheta}(\Phi(x)) + G^{\alpha}(\Phi(x)) \Phi^{\beta}(x)\right) + |K^-| y \\ &\leq C_{55} \left(1 + y + G^{n-\varepsilon_0 \wedge \vartheta}(y) + G^{\alpha}(y) y^{\beta}\right), \quad \text{for } (x,y) \in \mathcal{D}, \end{aligned} \tag{1.144}$$

where $C_{55} > 0$ is a constant. However, in view of the assumption $\frac{\alpha}{1-\beta} < n$, if we choose any

$$\varepsilon_4 \in \left]0, \varepsilon_0 \wedge \vartheta \wedge \left(n - \frac{\alpha}{1-\beta}\right)\right[\quad \text{and} \quad C_5 \geq C_{53} \vee C_{54} \vee C_{55},$$

then we can see that (1.142)–(1.144) imply (1.111).

To show that w satisfies (1.52), we first observe that the positivity of A, B and the lower bound in Lemma 2 imply that

$$w(x,y) \geq -C_1(1+y), \quad \text{for all } (x,y) \in \overline{\mathcal{W}}. \tag{1.145}$$

This estimate and the definition of w in $\mathcal{I}$, provided by (1.109)–(1.110), imply

$$\begin{aligned} w(x,y) &\geq -(C_1 + K^+)G^{[-1]}(x) - C_1 \\ &\geq -(C_1 + K^+)C_4 x^{\alpha/(1-\beta)} - C_1, \quad \text{for all } (x,y) \in \mathcal{I}, \end{aligned} \tag{1.146}$$

the second inequality following thanks to (1.88) and (1.103). Also, if $y^* < \infty$, then (1.145) and the definition of w in $\mathcal{D}$, given by (1.110), imply

$$\begin{aligned} w(x,y) &\geq -C_1(1+\Phi(x)) - |K^-| \max\{y, \Phi(x)\} \\ &\geq -(C_1 + |K^-|)y - C_1. \end{aligned} \tag{1.147}$$

However, (1.145)–(1.147) establish (1.52).

With reference to the construction of w, we will show that w is C^2 if we prove that w_x, w_{xx} and w_{yy} are continuous along the free boundaries F and G. To this end, we calculate

$$\begin{aligned} w_x(x,y) &= w_x\left(x, G^{[-1]}(x)\right) + \left[w_y\left(x, G^{[-1]}(x)\right) - K^+\right] \frac{dG^{[-1]}(x)}{dx} \\ &= w_x\left(x, G^{[-1]}(x)\right), \quad \text{for } (x,y) \in \mathcal{I}, \end{aligned} \tag{1.148}$$

and

$$\begin{aligned} w_{xx}(x,y) &= w_{xx}\left(x, G^{[-1]}(x)\right) + w_{xy}\left(x, G^{[-1]}(x)\right) \frac{dG^{[-1]}(x)}{dx} \\ &= w_{xx}\left(x, G^{[-1]}(x)\right), \quad \text{for } (x,y) \in \mathcal{I}, \end{aligned} \tag{1.149}$$

the second equalities following thanks to (1.78) that have been among the requirements leading to the equations specifying the function G. However, these calculations and the structure of w provided by (1.109)–(1.110) show that w_x and w_{xx} are continuous along G.

Now, if $y^* > 0$ and $y \in [0, y^*] \cap \mathbb{R}$, we can use (1.77) and (1.85) to calculate

$$\begin{aligned}\lim_{x \uparrow G(y)} w_{yy}(x,y) &= A''(y)G^n(y) + R_{yy}(G(y),y) \\ &= \frac{G^{-1}(y)}{\sigma^2(n-m)}\left[G'(y)\left[H(G(y),y) - rK^+\right] + G^{m+1}(y)\int_0^{G(y)} s^{-m-1}H_y(s,y)\,ds\right] \\ &= 0, \end{aligned} \tag{1.150}$$

the last equality following thanks to (1.122). Also, if $y^* < \infty$ and $y > y^*$, we can use (1.77), (1.92) and (1.94) to calculate

$$\begin{aligned}\lim_{x \uparrow G(y)} w_{yy}(x,y) &= A''(y)G^n(y) + B''(y)G^m(y) + R_{yy}(G(y),y) \\ &= 0. \end{aligned} \tag{1.151}$$

However, combining (1.150) and (1.151) with the fact that $w_{yy}(x,y) = 0$, for $(x,y) \in \mathcal{I}$, we conclude that w_{yy} is continuous along G.

Showing that w_x, w_{xx} and w_{yy} are continuous along F involves similar arguments.

By construction, we will prove that w satisfies the HJB equation (1.47)–(1.48) if we show that

$$\sigma^2x^2w_{xx}(x,y) + bxw_x(x,y) - rw(x,y) + h(x,y) \le 0, \quad \text{for } (x,y) \in \mathcal{I}, \tag{1.152}$$

$$w_y(x,y) + K^- \ge 0, \quad \text{for } (x,y) \in \mathcal{I},\ y > 0, \tag{1.153}$$

$$w_y(x,y) - K^+ \le 0, \quad \text{for } (x,y) \in \mathcal{W}, \tag{1.154}$$

$$w_y(x,y) + K^- \ge 0, \quad \text{for } (x,y) \in \mathcal{W},\ y > 0, \tag{1.155}$$

and, if $\mathcal{D} \neq \emptyset$,

$$\sigma^2x^2w_{xx}(x,y) + bxw_x(x,y) - rw(x,y) + h(x,y) \le 0, \quad \text{for } (x,y) \in \mathcal{D}, \tag{1.156}$$

$$w_y(x,y) - K^+ \le 0, \quad \text{for } (x,y) \in \mathcal{D}. \tag{1.157}$$

It is straightforward to see that either of (1.153) or (1.157) is equivalent to $K^+ + K^- \geq 0$, which is true by assumption. Recalling that $H \equiv h_y$, we can easily verify that, since $y \leq G^{[-1]}(x)$, for all $(x,y) \in \mathcal{I}$, (1.148) and (1.149) imply that (1.152) is equivalent to

$$\int_y^{G^{-1}(x)} \left[H(x,u) - rK^+\right] du \geq 0, \quad \text{for } (x,y) \in \mathcal{I}.$$

However, this inequality follows immediately from the assumption that $H(x,\cdot)$ is strictly decreasing, for all x, and (1.87) together with the second inequality in (1.102). Similarly, we can show that, if $\mathcal{D} \neq \emptyset$, then (1.156) is equivalent to

$$\int_{F^{-1}(x)}^{y} \left[H(x,u) + rK^-\right] du \leq 0, \quad \text{for } (x,y) \in \mathcal{D},$$

which is true by virtue of the first inequality in (1.102) and the assumption that $H(x,\cdot)$ is strictly decreasing, for all x.

Now, suppose that $y^* < \infty$, and fix any $y > y^*$. Since $w_y(F(y),y) = -K^-$ and $w_y(G(y),y) = K^+$, we will prove that both of (1.154) and (1.155) are satisfied if we show that

$$w_{yx}(x,y) \geq 0, \quad \text{for all } x \in]F(y), G(y)[. \tag{1.158}$$

To this end, we consider the transformation of the independent variable $x > 0$ provided by $z = \ln x$, and we write $w(x,y) = u(\ln x, y)$ for some function $u = u(z,y)$. It follows that (1.158) is true if and only if

$$u_{yz}(z,y) \geq 0, \quad \text{for all } z \in]\ln F(y), \ln G(y)[. \tag{1.159}$$

Now, since $w = w(x,y)$ satisfies (1.75) for $x \in]F(y), G(y)[$, u_y satisfies

$$\sigma^2 u_{yzz}(z,y) + \left(b - \sigma^2\right) u_{yz}(z,y) - r u_y(z,y) + H(e^z, y) = 0, \quad \text{for } z \in]\ln F(y), \ln G(y)[.$$

Recalling that H_x is continuous and $H_x(\cdot, y) \geq 0$ (see Assumption 1), we can differentiate this equation with respect to z to obtain

$$\begin{aligned} \sigma^2 (u_{yz})_{zz}(z,y) + \left(b - \sigma^2\right) (u_{yz})_z(z,y) - r u_{yz}(z,y) &= -e^z H_x(e^z, y), \\ &\leq 0, \quad \text{for } z \in]\ln F(y), \ln G(y)[. \end{aligned}$$

This inequality and the maximum principle imply that $u_{yz}(\cdot, y)$ does not have a negative minimum in the interval $]\ln F(y), \ln G(y)[$, so

$$\begin{aligned}\inf_{z\in]\ln F(y),\ln G(y)[} u_{yz}(z,y) &\geq \min_{z=\ln F(y),\ln G(y)} 0 \wedge u_{yz}(z,y) \\ &= \min_{z=F(y),G(y)} 0 \wedge w_{yx}(x,y) \\ &= 0.\end{aligned}$$

However, this calculation implies (1.159).

To proceed further, fix any $y \in [0, y^*] \cap \mathbb{R}$. Using the definition of R in (1.77), the expression for $A'(y)$ provided by (1.85) and the fact that $G(y)$ satisfies (1.83), we can see that, if we define $\overline{u}(x,y) = w_y(x,y) - K^+$, then

$$\begin{aligned}\overline{u}_x(x,y) = \frac{1}{\sigma^2(n-m)} &\left[-mx^{m-1} \int_x^{G(y)} s^{-m-1} \left[H(s,y) - rK^+ \right] ds \right. \\ &\left. + nx^{n-1} \int_x^{G(y)} s^{-n-1} \left[H(s,y) - rK^+ \right] ds \right], \quad \text{for } x \in]0, G(y)[.\end{aligned}$$

This calculation and the assumption that $H(\cdot, y)$ is strictly increasing imply that $\overline{u}_x(x,y) = w_{yx}(x,y) > 0$, for all $x \in [x^\dagger(y), G(y)[$, where $x^\dagger(y) \in]0, G(y)[$ is the unique point such that $H\left(x^\dagger(y), y\right) - rK^+ = 0$ (see Lemma 6). This observation and the boundary condition $w_y\left(G(y), y\right) = K^+$ imply

$$w_y(x,y) - K^+ < 0, \quad \text{for all } x \in [x^\dagger(y), G(y)[. \tag{1.160}$$

Furthermore, since

$$\sigma^2 x^2 \overline{u}_{xx}(x,y) + bx\overline{u}_x(x,y) - r\overline{u}(x,y) = -\left[H(x,y) - rK^+\right] \geq 0, \quad \text{for } x \in]0, x^\dagger(y)[,$$

the maximum principle implies that the function $x \mapsto \overline{u}(x,y) = w_y(x,y) - K^+$ has no positive maximum in the interval $]0, x^\dagger(y)[$, so

$$\sup_{x\in]0,x^\dagger(y)[} \left[w_y(x,y) - K^+\right] \leq \max_{x=0,x^\dagger(y)} 0 \vee \left[w_y(x,y) - K^+\right] = 0, \tag{1.161}$$

the equality following thanks to (1.160) and the fact that

$$\lim_{x\downarrow 0} w_y(x,y) = \lim_{x\downarrow 0} R_y(x,y) = \lim_{x\downarrow 0} \frac{H(x,y)}{r} \in [-K^-, K^+[. \tag{1.162}$$

The second equality here holds true because of (1.18), while the inclusion follows from the context (see Lemmas 6 and 7). However, (1.160) and (1.161) establish (1.154). Finally, if we define $\underline{u}(x,y) = w_y(x,y) + K^-$, then (1.162) and the assumption that $H(\cdot, y)$ is increasing imply

$$\sigma^2 x^2 \underline{u}_{xx}(x,y) + bx\underline{u}_x(x,y) - r\underline{u}(x,y) = -\left[H(x,y) + rK^-\right] \le 0, \quad \text{for all } x \in]0, G(y)[.$$

This calculation and the maximum principle imply that the function $x \mapsto \underline{u}(x,y) = w_y(x,y) + K^-$ has no negative minimum inside $]0, G(y)[$, so

$$\inf_{x \in]0,G(y)[} \left[w_y(x,y) + K^-\right] = \min_{x=0,G(y)} 0 \wedge \left[w_y(x,y) + K^-\right],$$

which combined with (1.162) and the boundary condition $w_y(G(y),y) + K^- = K^+ + K^- > 0$, proves (1.155), and the proof is complete. □

Chapter 2

Irreversible capacity expansion with proportional and fixed costs

2.1 Introduction

In this chapter, we aim to establish in a dynamical way the optimal capacity level of a given investment project, the operation of which is associated with random cash flows. The project's capacity level can only be increased, and each capacity increase is associated with a fixed plus a proportional cost. In parallel to Chapter 1, the investment project yields payoff at a rate that is dependent on its installed capacity level and on an underlying economic indicator such as the price of or the demand for the project's unique output commodity, which we model by a geometric Brownian motion. We consider a performance criterion which is equivalent to the expected, discounted payoff net of investment costs over an infinite time horizon. The objective is to determine the capacity expansion strategy that maximises this performance index. The resulting optimisation problem takes the form of a two-dimensional impulse control problem that we solve explicitly.

The chapter is organised as follows. Section 2.2 is concerned with a rigorous formulation of the investment decision model that we study. In Section 2.3, we derive sufficient conditions, which conform with economic intuition, for the associated optimisation problem to possess a finite value function, and we establish a number of estimates that we use in our subsequent analysis. Finally, we solve the optimisation problem considered in Section 2.4.

2.2 Problem formulation

We fix a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_t)$ satisfying the usual conditions of right continuity and augmentation by P-negligible sets, and carrying a standard, one-dimensional $(\mathcal{F}_t)$-Brownian motion W. We denote by $\mathcal{A}$ the family of all càglàd, $(\mathcal{F}_t)$-adapted, increasing and piecewise constant processes Z such that $Z_0 = 0$.

We consider an investment project that produces a given commodity, and we assume that the project's capacity, namely its rate of output, can be increased at any given time and by any amount. We denote by Y_t the project's capacity at time t, and we model capacity increases by jumps of an impulse control process $Z \in \mathcal{A}$, i.e. every time Z_t jumps, there is a capacity increase and $\Delta Z_t = \Delta Y_t$ is the size of the jump at time t. The capacity process Y is therefore given by

$$Y_t = y + Z_t, \quad Y_0 = y \geq 0, \tag{2.1}$$

where $y \geq 0$ is the project's initial capacity. Every process $Z \in \mathcal{A}$ is characterised by the collection $\mathcal{Z} = (\tau_1, \tau_2, \ldots, \tau_n; \ldots; \Delta Z_{\tau_1}, \Delta Z_{\tau_2}, \ldots, \Delta Z_{\tau_n}, \ldots)$ where τ_n is the $(\mathcal{F}_t)$-stopping time at which the n-th jump of Z occurs, while ΔZ_{τ_n} is the associated jump size. If the project's management adopts the capacity expansion strategy modelled by Z, then the project's capacity is increased at the times τ_n, $n \geq 1$, by an amount $\Delta Y_t = \Delta Z_t > 0$.

We assume that all randomness associated with the project's operation can be captured by a state process X that satisfies the SDE

$$dX_t = bX_t\,dt + \sqrt{2}\sigma X_t\,dW_t, \quad X_0 = x > 0, \tag{2.2}$$

for some constants b and σ. In practice, X_t can be an economic indicator reflecting, e.g., the value of one unit of the output commodity or the output commodity's demand or both, at time t.

To simplify the notation, we define

$$\mathcal{S} = \left\{(x,y) \in \mathbb{R}^2 : \ x > 0, \ y \geq 0\right\},$$

so that $\mathcal{S}$ is the set of all possible initial conditions. With each decision policy Z we associate the performance criterion

$$J_{x,y}(Z) = E\left[\int_0^\infty e^{-rt} h(X_t, Y_t)\, dt - \sum_{0 \leq t} e^{-rt}\left(K\Delta Z_t + c\right) \mathbf{1}_{\{\Delta Z_t > 0\}}\right], \tag{2.3}$$

where $h : \mathcal{S} \to \mathbb{R}$ is a given function and

$$r, K, c > 0 \tag{2.4}$$

are constants. Here, h models the running payoff resulting from the project's operation, and K, c provide a proportional and a fixed cost incurred each time that the project's capacity level is changed.

As it stands in (2.3), the performance index $J_{x,y}$ is not necessarily well-defined because the random variable inside the expectation may not be integrable or even well-defined. To address this issue, we define

$$U_T = \int_0^T e^{-rt} h(X_t, Y_t)\, dt - \sum_{0 \leq t \leq T} e^{-rt}\left(K\Delta Z_t + c\right) \mathbf{1}_{\{\Delta Z_t > 0\}}, \quad \text{for } T \geq 0. \tag{2.5}$$

In the next section (see Lemma 14, in particular), we are going to impose assumptions on h such that U_T is well-defined, for all $T > 0$, and *either*

$$U_\infty = \lim_{T \to \infty} U_T \text{ exists in } \mathbb{R}, \ P\text{-a.s.}, \quad \text{and} \quad U_\infty \in L^1(\Omega, \mathcal{F}, P), \tag{2.6}$$

in which case, we naturally define

$$J_{x,y}(Z) = E\left[U_\infty\right], \tag{2.7}$$

as in (2.3), *or* there exists an $(\mathcal{F}_t)$-adapted process V such that

$$U_T \leq V_T, \ \text{for all } T \geq 0, \quad \text{and} \quad \limsup_{T \to \infty} E\left[V_T\right] = -\infty, \tag{2.8}$$

in which case, we define

$$J_{x,y}(Z) = -\infty. \tag{2.9}$$

The objective is to maximise this performance index over all admissible capacity expansion strategies $Z \in \mathcal{A}$. The value function of the resulting optimisation problem is defined by

$$v(x, y) = \sup_{Z \in \mathcal{A}} J_{x,y}(Z). \tag{2.10}$$

2.3 Assumptions and preliminary estimates

The purpose of this section is to establish conditions on the problem's data under which our control problem is well-posed and its value function is finite, and to prove certain estimates that will be used in our analysis. We note that some of the results from Chapter 1 are reproduced here in order to make the presentation of this chapter self-contained. We first discuss an ODE that will play an instrumental role in the solution of the control problem considered.

Let $k :]0, \infty[\to \mathbb{R}$ be any measurable function such that

$$E\left[\int_0^\infty e^{-rt} |k(X_t)| \, dt\right] < \infty, \quad \text{for all } x > 0. \tag{2.11}$$

In view of the results in Proposition 4.1 of Knudsen, Meister and Zervos [KMZ98], the function $R(\cdot; k) :]0, \infty[\to \mathbb{R}$ given by

$$R(x; k) = \frac{1}{\sigma^2(n-m)} \left[x^m \int_0^x s^{-m-1} k(s) \, ds + x^n \int_x^\infty s^{-n-1} k(s) \, ds\right] \tag{2.12}$$

is well-defined and

$$R(x; k) = E\left[\int_0^\infty e^{-rt} k(X_t) \, dt\right] \tag{2.13}$$

Here, the constants $m < 0 < n$ are the solutions of the quadratic equation

$$\sigma^2 \lambda^2 + (b - \sigma^2)\lambda - r = 0, \tag{2.14}$$

given by

$$m, n = \frac{-(b - \sigma^2) \pm \sqrt{(b - \sigma^2)^2 + 4\sigma^2 r}}{2\sigma^2}. \tag{2.15}$$

Moreover, condition (2.11) is also sufficient for every solution of the ODE

$$\sigma^2 x^2 u''(x) + bxu'(x) - ru(x) + k(x) = 0. \tag{2.16}$$

can be expressed by

$$u(x) = Ax^n + Bx^m + R(x;k), \tag{2.17}$$

for some $A, B \in \mathbb{R}$.

With regard to $R(\cdot;k)$,

$$\text{if } k \text{ is increasing, then } R(\cdot;k) \text{ is increasing,} \tag{2.18}$$

and

$$\inf_{x>0} k(x) \geq 0 \quad \Leftrightarrow \quad \inf_{x>0} R(x;k) \geq 0. \tag{2.19}$$

For future reference, note that, given $\lambda \in \mathbb{R}$,

$$\begin{aligned} E\left[\int_0^\infty e^{-rt} X_t^\lambda \, dt\right] &= x^\lambda \int_0^\infty e^{\left[\sigma^2\lambda^2 + (b-\sigma^2)\lambda - r\right]t} E\left[e^{-\sigma^2\lambda^2 t + \sqrt{2}\sigma\lambda W_t}\right] dt \\ &= \begin{cases} \infty, & \text{if } \lambda \leq m \text{ or } \lambda \geq n, \\ -x^\lambda / \left[\sigma^2\lambda^2 + (b-\sigma^2)\lambda - r\right], & \text{if } \lambda \in]m, n[. \end{cases} \end{aligned} \tag{2.20}$$

We are going to need the following estimate that is related with the definitions above.

Lemma 11 *Given any $\lambda \in]0, n[$, there exist constants $\varepsilon_1, \varepsilon_2 > 0$ such that*

$$E\left[e^{-rt} \bar{X}_t^\lambda\right] \leq \frac{\sigma^2\lambda^2 + \varepsilon_2}{\varepsilon_2} x^\lambda e^{-\varepsilon_1 t} \quad \text{and} \quad E\left[\sup_{t\geq 0} e^{-rt} \bar{X}_t^\lambda\right] \leq \frac{\sigma^2\lambda^2 + \varepsilon_2}{\varepsilon_2} x^\lambda,$$

where $\bar{X}_t = \sup_{s\leq t} X_s$.

Proof. Since n is the positive solution of the quadratic equation (2.14), it follows that there exist $\varepsilon_1, \varepsilon_2 > 0$ such that

$$r - \varepsilon_1 > 0 \quad \text{and} \quad \sigma^2\lambda^2 + (b - \sigma^2)\lambda - (r - \varepsilon_1) = -\varepsilon_2.$$

Given such parameters, we define

$$\Psi = \sup_{t \geq 0} \left[-\frac{\sigma^2 \lambda^2 + \varepsilon_2}{\sqrt{2}|\sigma|\lambda} t + W_t \right],$$

we calculate

$$\begin{aligned}
e^{-rt} \bar{X}_t^\lambda &= x^\lambda e^{-\varepsilon_1 t} e^{-(r-\varepsilon_1)t} \sup_{s \leq t} \exp\left((r - \varepsilon_1)s - (\sigma^2 \lambda^2 + \varepsilon_2)s + \sqrt{2}\sigma\lambda W_s \right) \\
&= x^\lambda e^{-\varepsilon_1 t} \sup_{s \leq t} \left[\exp\left(-(r - \varepsilon_1)(t - s) \right) \exp\left(-(\sigma^2 \lambda^2 + \varepsilon_2)s + \sqrt{2}\sigma\lambda W_s \right) \right] \\
&\leq x^\lambda e^{-\varepsilon_1 t} e^{\sqrt{2}|\sigma|\lambda\Psi},
\end{aligned}$$

and we observe that

$$\sup_{t \geq 0} e^{-rt} \bar{X}_t^\lambda \leq x^\lambda e^{\sqrt{2}|\sigma|\lambda\Psi}.$$

Since Ψ is exponentially distributed with parameter $2\left(\sigma^2\lambda^2 + \varepsilon_2\right) / \left(\sqrt{2}|\sigma|\lambda\right)$ (see Karatzas and Shreve [KS88, Exercise 3.5.9], the two bounds follow by a simple integration. □

The following assumption ensures that the control problem formulated in Section 2.2 is well-posed and its value function is finite and identifies with an appropriate solution of the associated Hamilton-Jacobi-Bellman equation.

Assumption 3 The problem's data satisfy the following conditions:

(a) The function h is C^3, and

$$h_x(x, y) \geq 0, \text{ for all } y \geq 0. \tag{2.21}$$

If we define

$$H(x, y) = h_y(x, y), \quad (x, y) \in \mathcal{S}, \tag{2.22}$$

then, given any $y > 0$,

$$H_x(x, y) > 0, \text{ for all } x > 0, \quad \text{and} \quad \lim_{x \to \infty} H(x, y) = \infty, \tag{2.23}$$

and, given any $x > 0$,

$$H_y(x, y) < 0, \text{ for all } y > 0. \tag{2.24}$$

(b) The constants r, K, C are strictly positive (see (2.4)), and there exist constants

$$\alpha > 0,\ \beta \in]0,1[,\ \vartheta_1 \in]0,n[,\ \vartheta_2 \in]0,K] \text{ and } C > 0,$$

where $n > 0$ is given by (2.15), such that

$$\frac{\alpha}{1-\beta} \in]0,n[, \tag{2.25}$$

and we note that

$$\frac{\alpha}{1-\beta} \in]0,n[\quad \Leftrightarrow \quad \frac{n\beta}{n-\alpha} \in]0,1[. \tag{2.26}$$

$$-C(1+y) \leq h(x,y) \leq C(1+x^{n-\vartheta_1}) + Cx^\alpha y^\beta + r(K-\vartheta_2)y, \quad \text{for all } (x,y) \in \mathcal{S}. \tag{2.27}$$

$$-C \leq H(x,y) \leq \beta C x^\alpha y^{-(1-\beta)} + r(K-\vartheta_2), \quad \text{for all } (x,y) \in \mathcal{S}. \tag{2.28}$$

(c) There exist constants $y_1 > 0$ and Λ such that

$$\Lambda > \frac{K}{\vartheta_2}\frac{n\beta}{n-\alpha}C, \tag{2.29}$$

where α, β, ϑ_2, C are as in (b) above, and

$$H(x,y) \geq \beta\Lambda x^\alpha y^{-(1-\beta)}, \quad \text{for all } x > 0 \text{ and } y \geq y_1. \tag{2.30}$$

(d) Given any $y > 0$,

$$\int_0^x s^{-m-1}\left|H_y(s,y)\right| ds + \int_x^\infty s^{-n-1}\left|H_y(s,y)\right| ds < \infty.$$

□

Some of the conditions appearing in this assumption have a natural economical interpretation. Indeed, we can think of $H(x,y)\Delta y$ as the *additional* running payoff that we are faced with if we increase the project's capacity level from y to $y+\Delta y$, for small Δy, and the underlying state process X assumes the value x. In view of this observation, (2.23) reflects the idea that, given y, a small amount of extra capacity should be associated with increasing values of additional running payoff as the value of x, which, e.g., models the price of or the demand for the project's output commodity, is increasing. Similarly, (2.24) reflects the fact that, for a given value x of the underlying state process, the extra running payoff resulting from a small amount of additional capacity is decreasing as the level of the already installed capacity y increases. Also, (2.21) admits a similar, but simpler to express, economical interpretation.

The rest of the assumptions are of a technical nature. However, some of them cannot be significantly relaxed without losing the well-posedness of our control problem (see Lemma 13 below). Also, it is worth noting that part (c) of the assumption is rather weak because it only involves the tail of the function H as y tends to ∞; indeed, y_1 can be chosen arbitrarily large.

Example 2 A choice for the running payoff function h that has been widely considered in the economics literature is the so-called Cobb-Douglas production function given by

$$h(x,y) = x^{\alpha} y^{\beta}, \quad \text{for some constants } \alpha > 0 \text{ and } \beta \in]0,1[. \tag{2.31}$$

It is straightforward to verify that this choice for the running payoff function h satisfies all of our assumptions if and only if the parameters α and β appearing in (2.31) satisfy the inequality (2.25). To this end, it suffices to take $\vartheta_2 = K$, $C = 1$, and any $\Lambda \in \left]\frac{n\beta}{n-\alpha}, 1\right]$. □

It is a straightforward exercise to show that the bounds in (2.27)–(2.28) imply the following estimates.

Lemma 12 *With reference to the notation in (2.12), let $R^{[h]}, R^{[H]} : \mathcal{S} \to \mathbb{R}$ be the functions defined by $R^{[h]}(x,y) = R(x; h(\cdot,y))$, $R^{[H]}(x,y) = R(x; H(\cdot,y))$, respectively. The bounds provided by (2.27) and (2.28) in Assumption 3 imply that there exists a constant $C_1 > 0$ such that*

$$-C_1(1+y) \le R^{[h]}(x,y) \le C_1\left(1 + y + x^{n-\vartheta_1} + x^{\alpha}y^{\beta}\right), \quad \text{for all } (x,y) \in \mathcal{S},$$
$$-C_1 \le R^{[H]}(x,y) \le C_1\left(1 + x^{\alpha}y^{-(1-\beta)}\right), \quad \text{for all } (x,y) \in \mathcal{S}.$$

A bound such as the one in (2.27)–(2.28) is essential for the value function to be finite. Indeed, we can prove the following result.

Lemma 13 *Consider the control problem formulated in Section 2.2 that arises if the running payoff function h is the Cobb-Douglas production function defined in Example 2, and suppose that $\frac{\alpha}{1-\beta} > n > \alpha$ and $r > b$. Then, under any well-posed definition of the performance index $J_{x,y}$ that is consistent with (2.3), $v(x,y) = \infty$, for every initial condition $(x,y) \in \mathcal{S}$.*

Proof. Define $\lambda = \frac{n-\alpha}{\beta} > 0$ and note that the assumption that $\frac{\alpha}{1-\beta} > n$ implies that $\lambda < n$. Consider the capacity expansion strategy defined by

$$\tilde{Z}_t = \sum_{j=1}^{\infty} 2^{\lambda j} \mathbf{1}_{\{\bar{X}_t \in [2^{j-1}-1, 2^j-1[\}}, \quad \text{for } t \geq 0, \ j = 1, 2, \ldots, \tag{2.32}$$

where $\bar{X}_t = \sup_{s \leq t} X_s$, and note that the associated capacity level process satisfies

$$\begin{aligned}
\tilde{Y}_t^{\beta} \mathbf{1}_{\{\bar{X}_t \in [2^{j-1}-1, 2^j-1[\}} &= \left[y + 2^{\lambda j}\right]^{\beta} \mathbf{1}_{\{\bar{X}_t \in [2^{j-1}-1, 2^j-1[\}} \\
&\geq \left[y + (\bar{X}_t + 1)^{\lambda}\right]^{\beta} \mathbf{1}_{\{\bar{X}_t \in [2^{j-1}-1, 2^j-1[\}} \\
&\geq X_t^{n-\alpha} \mathbf{1}_{\{\bar{X}_t \in [2^{j-1}-1, 2^j-1[\}}.
\end{aligned}$$

With reference to (2.20), it follows that

$$E\left[\int_0^{\infty} e^{-rt} X_t^{\alpha} \tilde{Y}_t^{\beta} \, dt\right] \geq E\left[\int_0^{\infty} e^{-rt} X_t^{n} \, dt\right] = \infty. \tag{2.33}$$

To proceed further, we define the sequence of stopping times

$$\begin{aligned}
\tau_j &= \inf\left\{t \geq 0 : \ X_t \geq 2^j\right\}, \\
&= \inf\left\{t \geq 0 : \ \frac{b-\sigma^2}{\sqrt{2}|\sigma|} + \frac{\sigma}{|\sigma|} W_t \geq \frac{1}{\sqrt{2}|\sigma|} \ln\left(\frac{2^j}{x}\right)\right\}, \quad \text{for } j = 1, 2, \ldots.
\end{aligned}$$

Since the process $\left(\frac{\sigma}{|\sigma|} W_t, \ t \geq 0\right)$ is a standard Brownian motion, we can use the result of Exercise 3.5.10 in Karatzas and Shreve [KS88] and the definition of $n > 0$ given by (2.15) to calculate

$$\begin{aligned}
E[e^{-r\tau_j}] &= \exp\left(\frac{b-\sigma^2}{\sqrt{2}|\sigma|} \frac{1}{\sqrt{2}|\sigma|} \ln\left(\frac{2^j}{x}\right) - \frac{1}{\sqrt{2}|\sigma|} \ln\left(\frac{2^j}{x}\right) \sqrt{\frac{(b-\sigma^2)^2}{2\sigma^2} + 2}\right) \\
&= \left(\frac{x}{2^j}\right)^n.
\end{aligned} \tag{2.34}$$

In view of this calculation, we can see that

$$
\begin{aligned}
E\Bigg[\int_0^\infty & e^{-rt}(K\Delta\tilde{Z}_t + c)\mathbf{1}_{\{\Delta\tilde{Z}_t>0\}}\Bigg] \\
&= K\left(2^\lambda - 1\right) + c + E\left[\sum_{j=1}^\infty e^{-r\tau_j}\left[K\left(2^{\lambda(j+1)} - 2^{\lambda j}\right) + c\right]\right] \\
&\le K2^\lambda + c + \sum_{j=1}^\infty \left[K\left(2^\lambda - 1\right)2^{\lambda j} + c\right] E\left[e^{-r\tau_j}\right] \\
&= K2^\lambda + c + K\left(2^\lambda - 1\right)x^n \sum_{j=1}^\infty \left(\frac{1}{2^{n-\lambda}}\right)^j + cx^n \sum_{j=1}^\infty \left(\frac{1}{2^n}\right)^j \\
&< \infty, \qquad (2.35)
\end{aligned}
$$

the last equality following thanks to (2.34) and the inequality being true because $n-\lambda > 0$. However, combining this result with (2.33), we can see that

$$
E\left[\int_0^\infty e^{-rt}h(X_t,\tilde{Y}_t)\,dt - \sum_{0\le t} e^{-rt}\left(K\Delta\tilde{Z}_t + c\right)\mathbf{1}_{\{\Delta\tilde{Z}_t>0\}}\right]
$$

is well-defined and infinite, so, $J_{x,y}(\tilde{Z}) = \infty$, and the proof is complete. □

We can now prove that our assumptions are sufficient for the optimisation problem considered to be well-posed and for its value function to be finite.

Lemma 14 *Suppose that the running payoff function h satisfies (2.27) in Assumption 3 and that (2.4) is true. Given any initial condition $(x,y) \in \mathcal{S}$, (2.6)–(2.9) provide a well-defined performance criterion $J_{x,y}$, and the following statements hold true:*

(a) *Given any capacity expansion strategy $Z \in \mathcal{A}$, $J_{x,y}(Z) \in \mathbb{R}$ if and only if*

$$
E\left[\int_0^\infty e^{-rt}Y_t\,dt + \sum_{0\le t} e^{-rt}(K\Delta Z_t + c)\mathbf{1}_{\{\Delta Z_t>0\}}\right] < \infty. \qquad (2.36)
$$

(b) *Condition (2.36) implies*

$$
\liminf_{T\to\infty} e^{-rT}E\left[Y_{T+}\right] = 0. \qquad (2.37)
$$

(c) $v(x,y) \in \mathbb{R}$.

Proof. Fix any initial condition $(x, y) \in \mathcal{S}$ and any strategy $Z \in \mathcal{A}$. Since Z is an increasing càglàd process with $Z_0 = 0$, we use the integration by parts formula and (2.1) to calculate

$$\begin{aligned}
-K \sum_{0 \leq t \leq T} e^{-rt} \Delta Z_t \mathbf{1}_{\{\Delta Z_t > 0\}} &= -K \int_{[0,T]} e^{-rt} \, dZ_t \\
&= -K \left[e^{-rT} Z_{T+} + r \int_0^T e^{-rt} Z_t \, dt \right] \\
&= -rK \int_0^T e^{-rt} Y_t \, dt - K e^{-rT} Y_{T+} + Ky. \qquad (2.38)
\end{aligned}$$

This inequality and (2.27) in Assumption 3, imply that the random variables U_T defined by (2.5) satisfy

$$\begin{aligned}
U_T &\leq Ky + \int_0^T e^{-rt} [h(X_t, Y_t) - rKY_t] \, dt \\
&\leq Ky + C \int_0^T \left(1 + X_t^{n - \vartheta_1}\right) dt - \hat{V}_T, \qquad (2.39)
\end{aligned}$$

where

$$\hat{V}_T = \int_0^T e^{-rt} \left[r\vartheta_2 Y_t - C X_t^\alpha Y_t^\beta \right] dt, \quad \text{for } T \geq 0.$$

With reference to (2.20),

$$\begin{aligned}
I_1(x) &:= E\left[C \int_0^\infty e^{-rt} \left(1 + X_t^{n-\vartheta_1}\right) dt \right] \\
&= \frac{C}{r} - \frac{C x^{n - \vartheta_1}}{\sigma^2 (n - \vartheta_1)^2 + (b - \sigma^2)(n - \vartheta_1) - r} \in \,]0, \infty[. \qquad (2.40)
\end{aligned}$$

Now, suppose that $Z \in \mathcal{A}$ is associated with

$$E\left[\int_0^\infty e^{-rt} Y_t \, dt \right] = \infty. \qquad (2.41)$$

With regard to (2.25) in Assumption 3 and (2.20), we observe that

$$I_2(x) := E\left[\int_0^\infty e^{-rt} X_t^{\alpha/(1-\beta)} \, dt \right] < \infty. \qquad (2.42)$$

Therefore, given any constant $\mu > 0$,

$$E\left[\int_0^\infty e^{-rt} Y_t \mathbf{1}_{\left\{Y_t < \mu X_t^{\alpha/(1-\beta)}\right\}} \, dt \right] \leq \mu I_2(x) < \infty. \qquad (2.43)$$

It follows that (2.41) is true if and only if

$$E\left[\int_0^\infty e^{-rt} Y_t \mathbf{1}_{\left\{Y_t \geq \mu X_t^{\alpha/(1-\beta)}\right\}} \, dt\right] = \infty. \tag{2.44}$$

Now, fix any $\mu > 0$ such that $r\vartheta_2 - C\mu^{-(1-\beta)} > 0$, where the constants $\vartheta_2, C > 0$ and $\beta \in]0,1[$ are as in Assumption 3, and note that

$$\begin{aligned} E\left[\hat{V}_T\right] \geq & - C\mu^\beta E\left[\int_0^T e^{-rt} X_t^{\alpha/(1-\beta)} \mathbf{1}_{\left\{Y_t < \mu X_t^{\alpha/(1-\beta)}\right\}} \, dt\right] \\ & + \left(r\vartheta_2 - C\mu^{-(1-\beta)}\right) E\left[\int_0^T e^{-rt} Y_t \mathbf{1}_{\left\{Y_t \geq \mu X_t^{\alpha/(1-\beta)}\right\}} \, dt\right]. \end{aligned}$$

In view of (2.43)–(2.44) and the monotone convergence theorem, the right hand side of this inequality converges to ∞, which implies that $\lim_{T\to\infty} E[\hat{V}_T] = \infty$. However, this conclusion, (2.39) and (2.40) imply that there exists a process V such that (2.8) is satisfied and, therefore, $J_{x,y}(Z) = -\infty$.

To proceed further, let us assume that

$$E\left[\int_0^\infty e^{-rt} Y_t \, dt\right] < \infty, \tag{2.45}$$

which is necessary for condition (2.36) to be satisfied. Since Y is a finite variation process, its sample paths can have at most countable discontinuities. Using Fubini's theorem, we can see that this observation and (2.45) imply

$$\int_0^\infty e^{-rt} E\left[Y_{t+}\right] dt = E\left[\int_0^\infty e^{-rt} Y_{t+} \, dt\right] = E\left[\int_0^\infty e^{-rt} Y_t \, dt\right] < \infty,$$

which proves that (2.36) implies (2.37).

Now, using Hölder's inequality, we calculate

$$E\left[\int_0^\infty e^{-rt} X_t^\alpha Y_t^\beta \, dt\right] \leq I_2^{1-\beta}(x) \left(E\left[\int_0^\infty e^{-rt} Y_t \, dt\right]\right)^\beta < \infty, \tag{2.46}$$

where $I_2(x)$ is given by (2.42). This inequality, (2.40), (2.45) and the bounds in (2.27) in Assumption 3 imply

$$\begin{aligned} E\left[\int_0^\infty e^{-rt} \left|h(X_t, Y_t)\right| dt\right] & \leq E\left[\int_0^\infty e^{-rt} \left[C\left(1 + X_t^{n-\vartheta_1} + X_t^\alpha Y_t^\beta\right) + r(K^+ - \vartheta_2) Y_t\right] dt\right] \\ & < \infty, \end{aligned}$$

which combined with the dominated convergence theorem, implies that

$$\lim_{T\to\infty} E\left[\int_0^T e^{-rt} h(X_t, Y_t)\, dt\right] = E\left[\int_0^\infty e^{-rt} h(X_t, Y_t)\, dt\right] \in \mathbb{R}. \tag{2.47}$$

This observation gives rise to two possibilities. The first one is associated with the inequality

$$E\left[\sum_{0\le t} e^{-rt} \left(K\Delta Z_t + c\right) \mathbf{1}_{\{\Delta Z_t > 0\}}\right] < \infty.$$

In this case, $\lim_{T\to\infty} U_T$ exists, P-a.s., and belongs to $L^1(\Omega, \mathcal{F}, P)$, so $J_{x,y}(\xi^+, \xi^-)$ is finite and is given by (2.7). The second possibility is associated with

$$\lim_{T\to\infty} E\left[\sum_{0\le t\le T} e^{-rt} \left(K\Delta Z_t + c\right) \mathbf{1}_{\{\Delta Z_t > 0\}}\right] \equiv E\left[\sum_{0\le t} e^{-rt} \left(K\Delta Z_t + c\right) \mathbf{1}_{\{\Delta Z_t > 0\}}\right] = \infty,$$

which, combined with (2.47), implies that $\lim_{T\to\infty} E[U_T] = -\infty$, so (2.8) is satisfied for $V = U$ and $J_{x,y}(Z) = -\infty$.

The analysis above establishes the well-posedness of the definition of $J_{x,y}$ given by (2.6)–(2.9) as well as parts (a) and (b) of the lemma. To prove part (c) of the lemma, we first note that the results presented in (2.11)–(2.13) and the bounds in Lemma 12 imply

$$R^{[h]}(x, y) = E\left[\int_0^\infty e^{-rt} h(X_t, y)\, dt\right] \in \mathbb{R}.$$

However, this shows that our performance criterion is finite for the strategy that involves no capacity changes at any time, which proves that $v(x, y) > -\infty$. To show that $v(x, y) < \infty$, consider any capacity expansion strategy $Z \in \mathcal{A}$ such that $J_{x,y}(Z) > -\infty$. With reference to (2.45) and (2.46),

$$\begin{aligned} & E\left[\int_0^\infty e^{-rt} \left[r\vartheta_2 Y_t - C X_t^\alpha Y_t^\beta\right] dt\right] \\ & \quad \ge r\vartheta_2 E\left[\int_0^\infty e^{-rt} Y_t\, dt\right] - C I_2^{1-\beta}(x) \left(E\left[\int_0^\infty e^{-rt} Y_t\, dt\right]\right)^\beta \\ & \quad \ge -\frac{(1-\beta) r\vartheta_2}{\beta} \left(\frac{\beta C}{r\vartheta_2}\right)^{1/(1-\beta)} I_2(x), \quad \text{for all } T > 0, \end{aligned} \tag{2.48}$$

the second inequality following because, given any constants $\kappa, \lambda > 0$ and $\beta \in]0, 1[$,

$$\kappa Q - \lambda Q^\beta \geq -\frac{(1-\beta)\kappa}{\beta}\left(\frac{\beta\lambda}{\kappa}\right)^{1/(1-\beta)}, \quad \text{for all } Q \geq 0,$$

in particular, for $Q = E\left[\int_0^\infty e^{-rt} Y_t \, dt\right]$. However, (2.39), (2.40) and (2.48) imply

$$J_{x,y}(\xi^+, \xi^-) \leq I_1(x) + K^+ y + \frac{(1-\beta) r \vartheta_2}{\beta}\left(\frac{\beta C}{r\vartheta_2}\right)^{1/(1-\beta)} I_2(x),$$

which proves that $v(x, y) < \infty$ because the right hand side of this inequality is finite and independent of Z. □

2.4 The solution to the control problem

We now construct an explicit solution to the control problem formulated in Section 2.2 by constructing an appropriate solution to the associated Hamilton-Jacobi-Bellman (HJB) equation. To this end, we expect that the value function can be identified with a solution to the HJB equation

$$\begin{aligned} \max\{&\sigma^2 x^2 w_{xx}(x, y) + bx w_x(x, y) - rw(x, y) + h(x, y), \\ &- w(x, y) - c + \sup_{z>0}[w(x, y+z) - Kz]\} = 0, \quad (x, y) \in \mathcal{S}. \end{aligned} \tag{2.49}$$

To get a qualitative feeling about the origins of this equation, observe that, at time 0, the project's management has two options. The first one is to wait for a short time Δt and then continue optimally. With respect to Bellman's principle of optimality, this option, which is not necessarily optimal, is associated with the inequality

$$v(x, y) \geq E\left[\int_0^{\Delta t} e^{-rt} h(X_t, y) \, dt + e^{-r\Delta t} v(X_{\Delta t}, y)\right].$$

Applying Itô's formula to the second term in the expectation, and dividing by Δt before letting $\Delta t \downarrow 0$, we obtain

$$\sigma^2 x^2 v_{xx}(x, y) + bx v_x(x, y) - rv(x, y) + h(x, y) \leq 0. \tag{2.50}$$

The second option is to increase capacity by $\Delta Z_0 = z > 0$, and then continue optimally. Since such a capacity increase is not necessarily optimal, this action is associated with the inequality

$$v(x,y) \geq v(x,y+z) - Kz - c,$$

which implies

$$\sup_{z>0} [v(x,y+z) - Kz] - v(x,y) - c \leq 0, \tag{2.51}$$

because $z > 0$ was arbitrary. Since these two are the only options available, we expect that, given any initial condition $(x,y) \in \mathcal{S}$, one of them should be optimal, so that one of the inequalities (2.50)–(2.51) should hold with equality. However, this observation and (2.50)–(2.51) suggest that the value function v should identify with a solution w to the HJB equation (2.49). Now, it turns out that the value function v is not C^2, so we need to consider the following definition.

Definition 1 A function $w : \mathcal{S} \to \mathbb{R}$ is a *classical* solution of the HJB equation (2.49) if w is $C^{2,1}$,

$$\sigma^2 x^2 w_{xx}(x,y) + bxw_x(x,y) - rw(x,y) + h(x,y) \leq 0, \quad \text{Lebesgue-a.e., for all } y > 0,$$
$$\sup_{z>0} [v(x,y+z) - Kz] - v(x,y) - c \leq 0, \quad \text{for all } (x,y) \in \mathcal{S},$$

and there exists a set $\mathcal{I} \subset \mathcal{S}$ such that

$$\sigma^2 x^2 w_{xx}(x,y) + bxw_x(x,y) - rw(x,y) + h(x,y) = 0,$$

is satisfied in the interior of $\mathcal{I}^c$, Lebesgue-a.e., for all $y > 0$, and

$$\sup_{z>0} [v(x,y+z) - Kz] - v(x,y) - c = 0, \quad \text{for all } (x,y) \in \mathcal{I}.$$

□

To proceed further, we conjecture that the optimal strategy is characterised by a point $x^\circ > 0$ and two strictly increasing functions $G_0, G_1 : [x^\circ, \infty[\to \mathbb{R}_+$, such that $G_0(x) < G_1(x)$, for all $x \geq x^\circ$, and $G_0(x^\circ) = 0$. The function G_0 separates the state space $\mathcal{S}$ into two regions, the wait region $\mathcal{W}$ and the investment region $\mathcal{I}$, while the

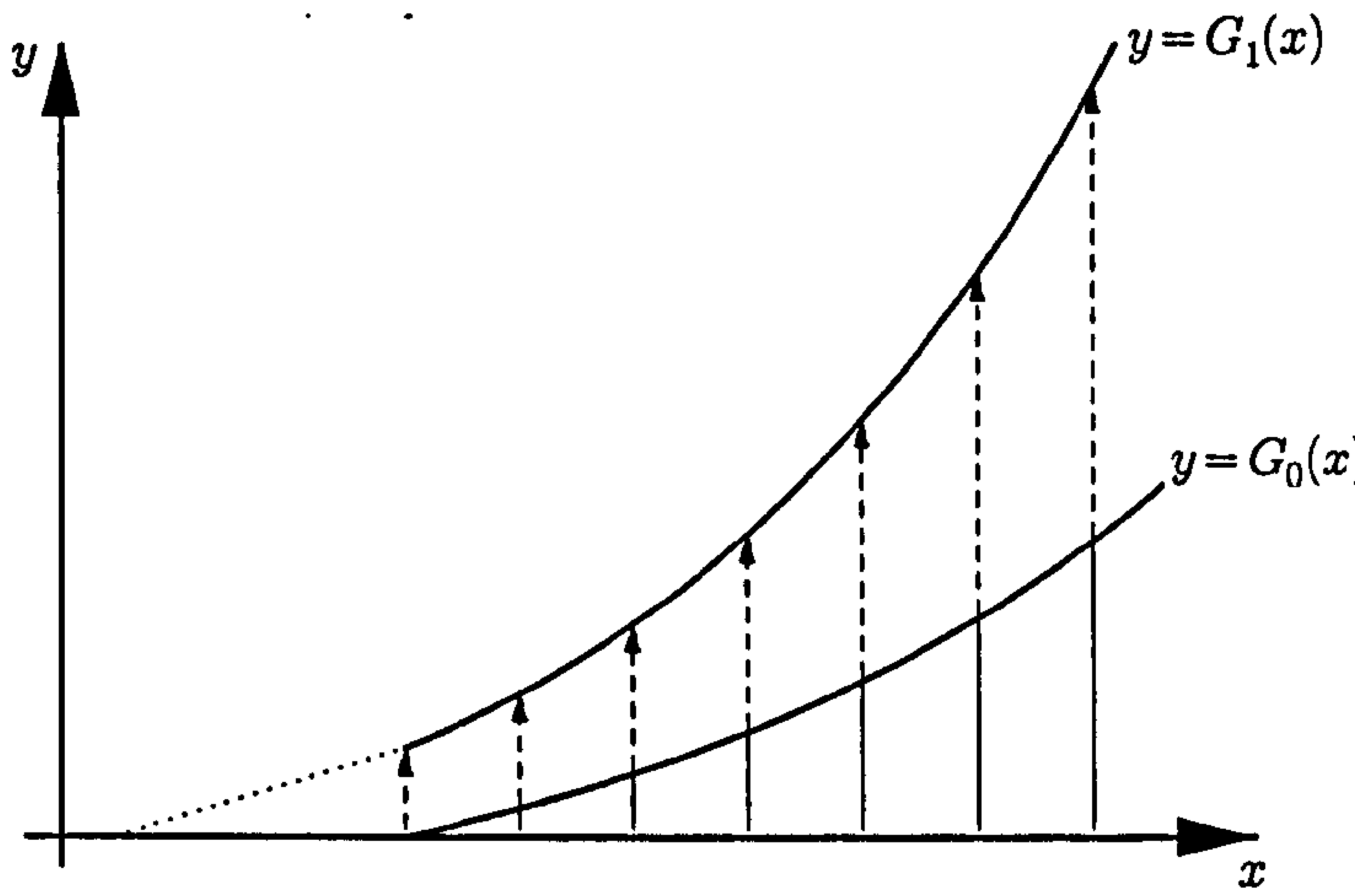

Figure 2.1: Illustration of a typical optimal capacity expansion strategy.

function G_1 provides the capacity level that should be reached whenever it is optimal to increase the project's capacity (see Figure 2.1).

With regard to the heuristic arguments considered above, we therefore look for a solution to the HJB equation (2.49) that satisfies

$$\sigma^2 x^2 w_{xx}(x,y) + bxw_x(x,y) - rw(x,y) + h(x,y) = 0, \quad \text{for } (x,y) \in \mathcal{W}, \tag{2.52}$$

and

$$w(x,y) = w(x, G_1(x)) - K\left[G_1(x) - y\right] - c, \quad \text{for } (x,y) \in \mathcal{I} \tag{2.53}$$

With regard to the discussion regarding the solvability of (2.16) in Section 3, every solution to equation (2.52) that remains bounded as $x \downarrow 0$ is given by

$$w(x,y) = A(y)x^n + R^{[h]}(x,y), \tag{2.54}$$

for some function A. Here,

$$R^{[h]}(x,y) = \frac{1}{\sigma^2(n-m)}\left[x^m \int_0^x s^{-m-1}h(s,y)\,ds + x^n \int_x^\infty s^{-n-1}h(s,y)\,ds\right]. \tag{2.55}$$

and the constants $m < 0 < n$ are given by (2.15).

To determine $A(y)$, $G_0(x)$ and $G_1(x)$, we postulate that $w(x,\cdot)$ is C^1 at the free boundary point $G_0(x)$, which yields

$$\lim_{u \downarrow G_0(x)} w_y(x,u) \equiv A'(G_0(x))x^n + R_y(x, G_0(x)) = K \equiv \lim_{u \uparrow G_0(x)} w_y(x,u). \tag{2.56}$$

Also, in view of the inequality

$$w(x, G_0(x) + z) - w(x, G_0(x)) - Kz - c \leq 0, \quad \text{for all } z > 0. \tag{2.57}$$

and the conjecture

$$w(x, G_1(x)) - w(x, G_0(x)) - K\left[G_1(x) - G_0(x)\right] = c, \tag{2.58}$$

we can see that the function

$$z \mapsto w(x, G_0(x) + z) - w(x, G_0(x)) - Kz - c$$

has a local maximum at $z^* = G_1(x) - G_0(x)$, which is associated with the equation

$$w_y(x, G_1(x)) \equiv A'(G_1(x))x^n + R_y^{[h]}(x, G_1(x)) = K. \tag{2.59}$$

Now, given any $y \geq 0$, (2.56) is equivalent to

$$A'(y)[G_0^{-1}(y)]^n + R_y^{[h]}(G_0^{-1}(y), y) = K, \tag{2.60}$$

while (2.59) is equivalent to

$$A'(y)[G_1^{-1}(y)]^n + R_y^{[h]}(G_1^{-1}(y), y) = K. \tag{2.61}$$

Eliminating $A'(y)$ from these two equations, and using the equality $\sigma^2 mn = -r$ as well as the definition of $R^{[h]}$, which implies that $R_y^{[h]} = R^{[H]}$, we can see that the points $G_0^{-1}(y)$ and $G_1^{-1}(y)$ should satisfy

$$F(G_1^{-1}(y), y, G_0^{-1}(y)) = 0, \tag{2.62}$$

where

$$F(x, y, z) = z^{-n} R^{[H(\cdot)-rK]}(z, y) - x^{-n} R^{[H(\cdot)-rK]}(x, y), \tag{2.63}$$

and

$$R^{[H(\cdot)-rK]}(x, y) = \frac{1}{\sigma^2(n-m)}\left[x^m \int_0^x s^{-m-1}[H(s, y) - rK]\, ds \right.$$
$$\left. + x^n \int_x^\infty s^{-n-1}[H(s, y) - rK]\, ds\right]. \tag{2.64}$$

To proceed further, let us assume that G_0 and G_1 are C^1. In this case, we can differentiate (2.58) with respect to x, and use (2.56) and (2.59) to obtain

$$w_x(x, G_1(x)) = w_x(x, G_0(x)), \tag{2.65}$$

which in view of (2.54), implies

$$[A(G_1(x)) - A(G_0(x))]\, x^n = -\frac{x}{n}\left[R_x^{[h]}(x, G_1(x)) - R_x^{[h]}(x, G_0(x))\right]. \tag{2.66}$$

However, combining this with (2.58) and (2.54), we can see that $G_0(x)$ and $G_1(x)$ should satisfy

$$\begin{aligned} &-\frac{x}{n}\left[R_x^{[h]}(x, G_1(x)) - R_x^{[h]}(x, G_0(x))\right] \\ &\qquad + \left[R^{[h]}(x, G_1(x)) - R^{[h]}(x, G_0(x))\right] - K[G_1(x) - G_0(x)] - c = 0, \end{aligned} \tag{2.67}$$

which, in view of the definition (2.55) of R and the equality $\sigma^2 mn = -r$, is equivalent to

$$\Phi(x, G_0(x), G_1(x)) = 0, \tag{2.68}$$

where

$$\Phi(x, y, p) = \int_y^p x^m \int_0^x s^{-m-1}\left[H(s, u) - rK\right] ds\, du + \frac{rc}{m}. \tag{2.69}$$

To summarise the heuristic discussion above, suppose that we can find a point $x^\circ > 0$ and two strictly increasing functions $G_0, G_1 : [x^\circ, \infty[\to \mathbb{R}_+$ satisfying (2.62) and (2.68). Since F and Φ are C^1 in each of their arguments, both of G_0 and G_1 are C^1. With regard to (2.60) or (2.61), if we choose

$$\begin{aligned} A(y) = \frac{1}{\sigma^2(n-m)}\Bigg[\int_y^\infty (G_1^{[-1]}(u))^{m-n} \int_0^{G_1^{[-1]}(u)} s^{-m-1}[H(s, u) - rK]\, ds\, du \\ + \int_y^\infty \int_{G_1^{[-1]}(u)}^\infty s^{-n-1}[H(s, u) - rK]\, ds\, du\Bigg], \end{aligned} \tag{2.70}$$

then, assuming the integrals are well-defined and finite, (2.56) and (2.59) are true, which, along with the C^1 continuity of G_0 and G_1, imply that (2.66) is also satisfied.

Moreover, (2.68) implies that (2.67) is true, which, combined with (2.66), implies that (2.58) is satisfied as well. In light of these observations, we can see that constructing a solution w to the HJB equation (2.49) amounts to finding functions G_0 and G_1 satisfying (2.62) and (2.68).

The next result is concerned with this construction and the associated solution to the HJB equation (2.49).

Lemma 15 *Suppose that Assumption 3 holds. The system of equations (2.62) and (2.68) define a point $x^\circ > 0$ and two C^1, increasing functions $G_0, G_1 : [x^\circ, \infty[\to \mathbb{R}_+$ such that $G_0(x) < G_1(x)$, for all $x \geq x^\circ$, $G_0(x^\circ) = 0$, and*

$$G_1(x) \leq C_2 x^{\frac{\alpha}{1-\beta}}, \quad \text{for all } x \geq 0, \tag{2.71}$$

for some constant $C_2 > 0$. The function w defined by

$$w(x,y) = \begin{cases} A(y)x^n + R(x,y), & \text{for } (x,y) \text{ such that } y > G_0(x), \\ w(x, G_1(x)) - K\left[G_1(x) - y\right] - c, & \text{for } (x,y) \text{ such that } 0 \leq y \leq G_0(x), \end{cases} \tag{2.72}$$

where A is given by (2.70), is C^1, and, given any $y \geq 0$, $w(\cdot, y)$ is C^2 outside the graph of G_0. Also, w is a classical solution to the HJB equation (2.49), in the sense of Definition 1, and there exist constants $C_3 > 0$ and $\varepsilon_3 \in]0, n[$ such that

$$-C_3(1 + y + x^{\frac{\alpha}{1-\beta}}) \leq w(x,y) \leq C_3\left(1 + y + [G_1^{[-1]}(y)]^{n-\varepsilon_3} + [G_1^{[-1]}(y)]^\alpha y^\beta + x^{n-\varepsilon_3} + x^{\frac{\alpha}{1-\beta}}\right), \tag{2.73}$$

for all $(x,y) \in \mathcal{S}$.

We can now prove the main result of the paper.

Theorem 16 *Consider the capacity control problem formulated in Section 2.2, and suppose that Assumption 3 holds. The value function v is equal to the solution to the HJB equation (2.49) constructed in Lemma 15. Apart from an initial jump of size $(G_1(x) - y)^+$ at time 0, the optimal capacity level process Y° has jumps of size provided by the function $G_1 - G_0$ that occur at the $(\mathcal{F}_t)$-stopping times when the process (X, Y°) hits the graph of G_0, and is constructed rigorously in the proof below.*

Proof. Fix any initial condition (x, y) and any admissible strategy $Z \in \mathcal{A}$ such that $J_{x,y}(Z) > -\infty$. Since Y is piecewise constant and $w(\cdot, y)$ is C^2 outside the graph of G_0, for all $y \geq 0$, we can use Itô's formula and the fact that X has continuous sample paths to obtain

$$\begin{aligned} e^{-rT} w(X_T, Y_{T+}) = w(x, y) + \int_0^T e^{-rt} \left[\sigma^2 X_t^2 w_{xx}(X_t, Y_t) + b X_t w_x(X_t, Y_t) - r w(X_t, Y_t) \right] dt \\ + M_T + \sum_{0 \leq t \leq T} e^{-rt} \left[w(X_t, Y_{t+}) - w(X_t, Y_t) \right], \end{aligned}$$

where

$$M_T = \sqrt{2}\sigma \int_0^T e^{-rt} X_t w_x(X_t, Y_t)\, dW_t, \quad T \geq 0. \tag{2.74}$$

Recalling the definition of U in (2.5), this implies

$$\begin{aligned} & U_T + e^{-rT} w(X_T, Y_{T+}) \\ & = w(x, y) + \int_0^T e^{-rt} \left[\sigma^2 X_t^2 w_{xx}(X_t, Y_t) + b X_t w_x(X_t, Y_t) - r w(X_t, Y_t) + h(X_t, Y_t) \right] dt \\ & \quad + M_T + \sum_{0 \leq t \leq T} e^{-rt} \left[w(X_t, Y_t + \Delta Z_t) - w(X_t, Y_t) - K \Delta Z_t - c \right] \mathbf{1}_{\{\Delta Z_t > 0\}}. \end{aligned} \tag{2.75}$$

Since w satisfies the HJB equation (2.49), it follows that

$$U_T + e^{-rT} w(X_T, Y_{T+}) \leq w(x, y) + M_T. \tag{2.76}$$

Now, in view of (2.38) and the assumption $K > 0$,

$$-e^{-rT} Y_{T+} \geq - \sum_{0 \leq t \leq T} e^{-rt} \Delta Z_t \mathbf{1}_{\{\Delta Z_t > 0\}} - y,$$

which, combined with the lower bound in (2.73), yields

$$e^{-rT} w(X_T, Y_{T+}) \geq -C_3 \left(e^{-rT} + \sum_{0 \leq t \leq T} e^{-rt} \Delta Z_t \mathbf{1}_{\{\Delta Z_t > 0\}} + e^{-rT} X_T^{\frac{\alpha}{1-\beta}} \right). \tag{2.77}$$

Furthermore, using (2.27) in Assumption 3, we obtain

$$\int_0^T e^{-rt} h(X_t, Y_t)\, dt \geq -C \int_0^T e^{-rt} Y_t\, dt - \frac{C}{r} \left(1 - e^{-rT}\right). \tag{2.78}$$

Combining (2.77) and (2.78) with (2.76), we can see that

$$\inf_{T\geq 0} M_T \geq -C_4\left(1+\int_0^\infty e^{-rt}Y_t\,dt+\sum_{[0,\infty[} e^{-rt}\Delta Z_t \mathbf{1}_{\{\Delta Z_t>0\}}+\sup_{T\geq 0} e^{-rT}\bar{X}_T^{\frac{\alpha}{1-\beta}}\right), \tag{2.79}$$

where $C_4 > 0$ is a constant and $\bar{X}_t = \sup_{s\leq t} X_s$. Recalling the assumption that $\frac{\alpha}{1-\beta} \in]0, n[$, we can see that the second bound in Lemma 11 and (2.36) in Lemma 14 imply that the random variable on the right hand side of this inequality has finite expectation. It follows from Exercise IV(1.46) in [RY04] that the stochastic integral M defined by (2.74) is a supermartingale, and therefore, $E[M_T] \leq 0$, for all $T > 0$. Taking expectations in (2.76), we therefore obtain

$$E[U_T] \leq w(x,y) + \liminf_{T\to\infty} e^{-rT} E[-w(X_T, Y_{T+})]. \tag{2.80}$$

Furthermore, since

$$U_T \geq -C_4\left(1+\int_0^\infty e^{-rt}Y_t\,dt+\sum_{[0,\infty[} e^{-rt}\Delta Z_t \mathbf{1}_{\{\Delta Z_t>0\}}\right) = -N,$$

where N is a positive random variable with finite expectation. Then, applying Fatou's lemma to $U_T + N$ and simplifying, we obtain

$$J_{x,y}(Z) \leq \liminf_{T\to\infty} E[U_T], \tag{2.81}$$

while (2.73) implies

$$\begin{aligned}\liminf_{T\to\infty} e^{-rT} E[-w(X_T, Y_{T+})] &\leq \lim_{T\to\infty} e^{-rT} C_3 + C_3 \liminf_{T\to\infty} e^{-rT} E[Y_{T+}] \\ &\quad + C_3 \lim_{T\to\infty} e^{-rT} E\left[\bar{X}_T^{\alpha/(1-\beta)}\right] \\ &= 0, \end{aligned} \tag{2.82}$$

the equality being true thanks to the first bound in Lemma 11 and (2.36). However, (2.80)–(2.82) imply that $J_{x,y}(Z) \leq w(x,y)$, which proves that $v(x,y) \leq w(x,y)$.

Now, let us set

$$\tau_0 = 0 \quad \text{and} \quad Z_t^{(0)} = [G_1(x) - y]\,\mathbf{1}_{\{y<G_0(x)\}}, \tag{2.83}$$

and define iteratively the $(\mathcal{F}_t)$-stopping times τ_n and the processes $Z^{(n)}$ by

$$\tau_{k+1} = \inf\left\{t \geq \tau_k :\ X_t \geq G_0^{[-1]}\left(y + Z_t^{(k)}\right)\right\}, \quad \text{for } k = 0, 1, \ldots,$$

$$Z_t^{(k+1)} = Z_t^{(k)} + \left[G_1(X_{\tau_{k+1}}) - G_0(X_{\tau_{k+1}})\right] \mathbf{1}_{\{t > \tau_{k+1}\}}, \quad \text{for } k = 0, 1, \ldots. \tag{2.84}$$

Observing that $\lim_{k\to\infty} \tau_k = \infty$, P-a.s., and that $Z_t^{(k)} = Z_t^{(k+1)}$, for all $t \in [0, \tau_{k+1}]$, and $k \geq 0$, we define the capacity expansion process Z° by $Z_t^\circ = Z_t^{(k)}$ for $t < \tau_k$, and we note that the associated capacity process Y° satisfies

$$Y^\circ \leq y \mathbf{1}_{\left\{\bar{X}_t \leq G_0^{[-1]}(y)\right\}} + G_1(\bar{X}_t) \mathbf{1}_{\left\{\bar{X}_t > G_0^{[-1]}(y)\right\}}. \tag{2.85}$$

This inequality and the upper estimate of w in (2.73) in Lemma 15 imply

$$e^{-rT} w(X_T, Y_T^\circ) \leq C_{53}\, e^{-rT} \left(1 + \bar{X}_T^{\frac{\alpha}{1-\beta}} + \bar{X}_T^{n-\varepsilon}\right). \tag{2.86}$$

Also, this inequality and the upper bound on h in Assumption 3.(2.27), imply

$$\begin{aligned}\int_0^T e^{-rt} h(X_t, Y_t^\circ)\, dt &\leq C_{54} \left(1 + \int_0^\infty e^{-rt} \left[X_t^{n-\vartheta_1} + X_t^\alpha (Y_t^\circ)^\beta + Y_t^\circ\right] dt\right) \\ &\leq C_{55} \left(1 + \int_0^\infty e^{-rt} \left[\bar{X}_t^{n-\vartheta_1} + \bar{X}_t^{\alpha/(1-\beta)}\right] dt\right).\end{aligned} \tag{2.87}$$

However, these inequalities and the estimates in Lemma 11 imply

$$E\left[\sup_{T>0} \left(\int_0^T e^{-rt} h(X_t, Y_t^\circ)\, dt + e^{-rT} w(X_T, Y_T^\circ)\right)\right] < \infty. \tag{2.88}$$

Now, with regard to the construction of Y°, we can see that (2.75) implies

$$\begin{aligned}\int_0^T e^{-rt} h(X_t, Y_t^\circ)\, dt - \sum_{0 \leq t \leq T} e^{-rt} (K \Delta Z_t + c) \mathbf{1}_{\{\Delta Z_t > 0\}} \\ + e^{-rT} w(X_T, Y_{T+}^\circ) = w(x, y) + M_T^\circ,\end{aligned} \tag{2.89}$$

where M° is defined as in (2.74) with $Y_t = Y_t^\circ$. This identity and (2.88) imply that $E\left[\sup_{T>0} M_T^\circ\right] < \infty$, so the stochastic integral M is a submartingale by Exercise IV(1.46) in [RY04]. In view of this observation, we can take expectations in (2.89) and pass to the limit to obtain

$$\begin{aligned}J_{x,y}(Z^\circ) &\geq w(x, y) + \liminf_{T\to\infty} e^{-rT} E[-w(X_T, Y_T^\circ)] \\ &\geq w(x, y) + C_{56} \liminf_{T\to\infty} e^{-rT} E\left[1 + \bar{X}_t^{n-\vartheta_1} + \bar{X}_t^{\alpha/(1-\beta)}\right] \\ &= w(x, y).\end{aligned} \tag{2.90}$$

Here, the second inequality follows from the upper bound of w in (2.73), (2.85) and (2.71). However, combining with the inequality $v(x, y) \leq w(x, y)$ that we proved above, we deduce that $v(x, y) = w(x, y)$ and that Z° is optimal, and the proof is complete. $\square$

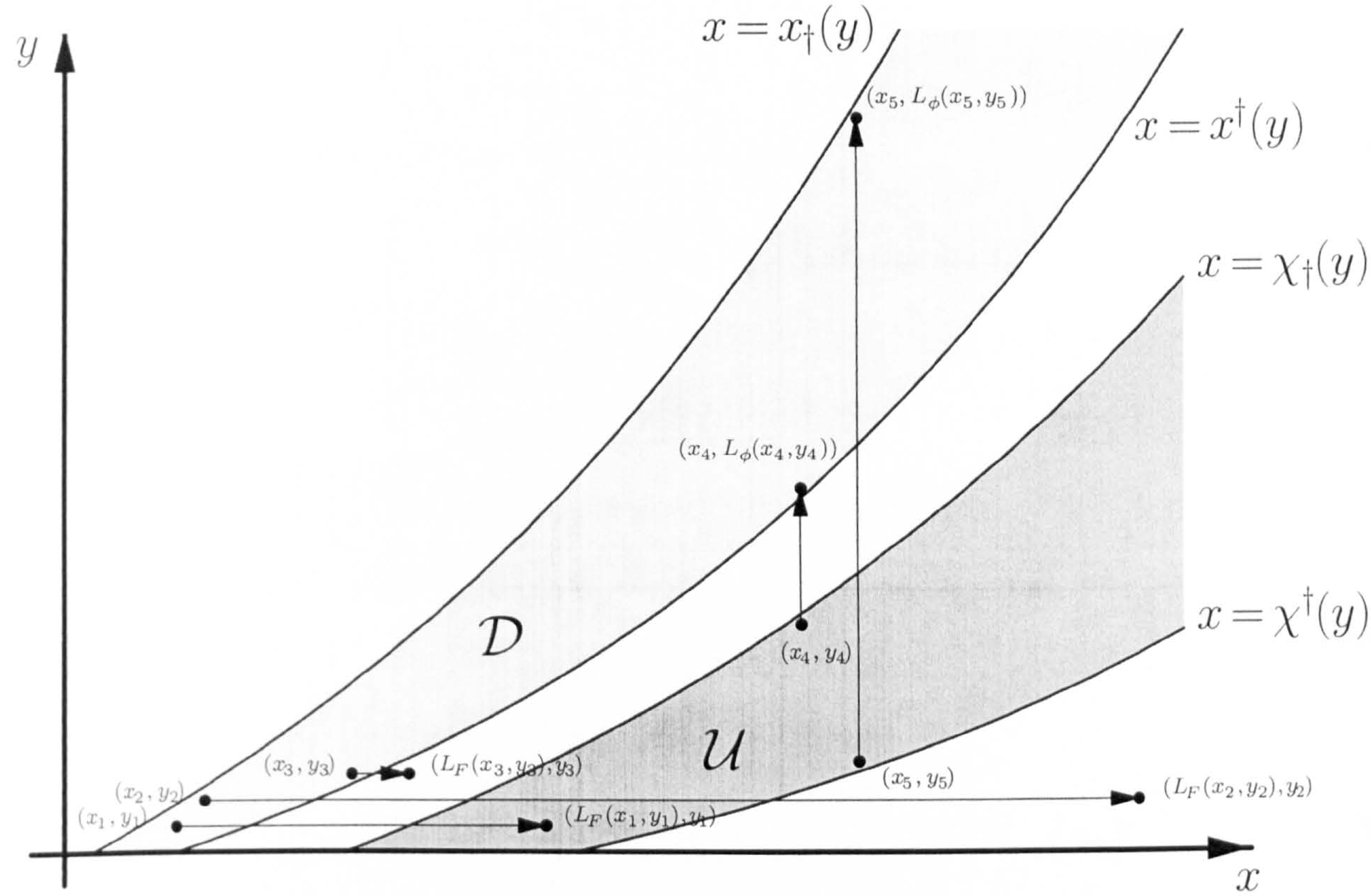

Figure 2.2: Illustration of the functions L_F and L_Φ constructed in Lemmas 17 and 18.

2.5 Appendix: Proof of Lemma 15

To establish Lemma 15, we first need to prove a number of preliminary results. The next one is concerned with a study of the function F defined by (2.62).

Lemma 17 *The equations*

$$q_\dagger(x, y) := -x^{m-n} \int_0^x s^{-m-1}[H(s, y) - rK]\, ds - \int_x^\infty s^{-n-1}[H(s, y) - rK]\, ds = 0, \tag{2.91}$$

$$q^\dagger(x, y) := \int_0^x s^{-m-1}[H(s, y) - rK]\, ds = 0, \tag{2.92}$$

define uniquely two strictly increasing, C^1 functions $x_\dagger, x^\dagger : [0, \infty[\to [0, \infty[$, respec-

tively, such that

$$\left[\frac{\sigma^2\vartheta_2}{\beta C}(\alpha-m)(n-\alpha)\right]^{\frac{1}{\alpha}} y^{\frac{1-\beta}{\alpha}} \le x_\dagger(y) < x^\dagger(y), \quad \textit{for all } y \ge 0. \tag{2.93}$$

Also, if we define

$$\mathcal{D} = \left\{(x,y) \in \mathcal{S} : \ x \in [x_\dagger(y), x^\dagger(y)]\right\}, \tag{2.94}$$

then the following statements are true:

(*a*) *For $(x,y) \in \mathcal{S}\backslash\mathcal{D}$, the equation $F(x,y,z) = 0$ has no solution $z > x$.*

(*b*) *There exists a unique mapping $L_F : \mathcal{D} \to [0,\infty[$ such that*

$$x < L_F(x,y) \quad \textit{and} \quad F(x,y,L_F(x,y)) = 0. \tag{2.95}$$

Moreover,

$$\frac{\partial}{\partial x}L_F(x,y) < 0 \quad \textit{and} \quad \frac{\partial}{\partial y}L_F(x,y) > 0, \quad \textit{for all } (x,y) \in \operatorname{int}\mathcal{D}, \tag{2.96}$$

$$\lim_{x\downarrow x_\dagger(y)} L_F(x,y) = \infty \quad \textit{and} \quad \lim_{x\uparrow x^\dagger(y)} L_F(x,y) = x^\dagger(y). \tag{2.97}$$

Proof. Consider (2.92), fix any $y > 0$, and observe that the upper bound in (2.28) in Assumption 3 implies that

$$\inf_{x>0} H(x,y) - rK = -r\vartheta_2 < 0. \tag{2.98}$$

Combining this inequality with (2.23) in Assumption 3, we can see that there exists a unique point $x_* = x_*(y) > 0$ such that

$$\frac{\partial}{\partial x}q^\dagger(x,y) = x^{-m-1}\left[H(x,y) - rK\right] \begin{cases} < 0, & \text{for all } x \in]0, x_*(y)[, \\ > 0, & \text{for all } x > x_*(y). \end{cases} \tag{2.99}$$

In view of this calculation, we combine the fact that $q^\dagger(\cdot,y)$ is strictly decreasing in $]0,x_*(y)[$ and strictly increasing in $]x_*(y),\infty[$, with $q^\dagger(0,y) = 0$, to see that $q^\dagger(x,y) < 0$, for all $x \le x_*(y)$, in particular, $q^\dagger(x_*(y),y) < 0$. Therefore, if $q^\dagger(x,y) = 0$ has a solution $x > 0$, then this must satisfy $x > x_*(y)$. Also, given that it exists, this solution is unique because $q^\dagger(\cdot, y)$ is strictly increasing in $]x_*(y), \infty[$. To prove that the required solution indeed exists, it suffices to show that $\lim_{x\to\infty} q^\dagger(x,y) = \infty$. The assumption that $\lim_{x\to\infty} H(x,y) = \infty$ implies that, given any constant $M > 0$, there exists $\beta > x_*(y)$ such that $H(x,y) - rK \geq M$, for all $x \geq \beta$. However, given any such choice of these constants, we calculate

$$\begin{aligned}
\lim_{x\to\infty} q^\dagger(x,y) &= \lim_{x\to\infty} \left[q^\dagger(\beta, y) + \int_\beta^x s^{-m-1} \left[H(s,y) - rK \right] ds \right] \\
&\geq \lim_{x\to\infty} \left[q^\dagger(\beta, y) + \frac{M}{m}\beta^{-m} - \frac{M}{m} x^{-m} \right] \\
&= \infty.
\end{aligned}$$

It follows that equation (2.92) defines uniquely a continuous function $x^\dagger :]0,\infty[\to]0,\infty[$ such that

$$\text{given any } y > 0, \quad q^\dagger(x,y) \begin{cases} < 0, & \text{for } x < x^\dagger(y), \\ = 0, & \text{for } x = x^\dagger(y), \\ > 0, & \text{for } x > x^\dagger(y). \end{cases} \tag{2.100}$$

Moreover, the arguments above imply that

$$H(x^\dagger(y), y) - rK > 0, \quad \text{for all } y > 0. \tag{2.101}$$

To see that $x^\dagger$ is C^1 and strictly increasing, we differentiate $q^\dagger(x^\dagger(y), y) = 0$ with respect to y to obtain

$$\begin{aligned}
\frac{\partial}{\partial y} x^\dagger(y) &= -\frac{1}{(x^\dagger(y))^{-m-1}[H(x^\dagger(y), y) - rK]} \int_0^{x^\dagger(y)} s^{-m-1} H_y(s,y)\, ds \\
&> 0,
\end{aligned} \tag{2.102}$$

the inequality following from (2.101) and (2.24) in Assumption 3. Since $x^\dagger :]0,\infty[\to]0,\infty[$ is increasing, we can extend its domain by defining $x^\dagger(0) = \lim_{y\downarrow 0} x^\dagger(y)$.

Now, fix any $y > 0$, consider (2.91) and observe that (2.92) and (2.100) imply

$$\begin{aligned}
q_\dagger(x^\dagger(y), y) &= -(x^\dagger(y))^{m-n} q^\dagger(x^\dagger(y), y) - \int_{x^\dagger(y)}^\infty s^{-n-1}[H(s,y) - rK]\, ds \\
&= -\int_{x^\dagger(y)}^\infty s^{-n-1}[H(s,y) - rK]\, ds \\
&< 0,
\end{aligned} \tag{2.103}$$

the inequality following thanks to (2.101) and the assumption that $H(\cdot, y)$ is strictly increasing. Also, note that

$$
\begin{aligned}
\frac{\partial}{\partial x} q_\dagger(x,y) &= (n-m)x^{m-n-1}\, q^\dagger(x,y) \\
&< 0, \quad \text{for all } x \in \,]0, x^\dagger(y)[. \qquad (2.104)
\end{aligned}
$$

To proceed further, we fix any ϵ, $x_\epsilon > 0$ such that $H(s,y) - rK < -\epsilon$, for all $s \leq x_\epsilon$. For such a choice of parameters, we obtain

$$
\begin{aligned}
\lim_{x \downarrow 0} q_\dagger(x,y) &\geq \lim_{x \downarrow 0} \epsilon \left[x^{m-n} \int_0^x s^{-m-1}\, ds + \int_x^{x_\epsilon} s^{-n-1}\, ds \right] - \int_{x_\epsilon}^\infty s^{-n-1} [H(s,y) - rK]\, ds \\
&= \infty. \qquad (2.105)
\end{aligned}
$$

However, combining this calculation with (2.103) and (2.104), we can see that, there exists a unique $x_\dagger(y) \in]0, x^\dagger(y)[$ such that $q_\dagger(x_\dagger(y), y) = 0$, for all $y > 0$. Moreover,

$$
q_\dagger(x,y) \begin{cases} > 0, & \text{for } x \in]0, x_\dagger(y)[, \\ < 0, & \text{for } x \in]x_\dagger(y), x^\dagger(y)[. \end{cases} \qquad (2.106)
$$

To see that $x_\dagger$ is C^1 and strictly increasing, we differentiate $q_\dagger(x_\dagger(y), y) = 0$ with respect to y to obtain

$$
\begin{aligned}
\frac{\partial}{\partial y} x_\dagger(y) &= (n-m)^{-1}\, x_\dagger(y)^{n-m+1}\, \frac{1}{q^\dagger(x_\dagger(y), y)} \\
&\quad \times \left[x_\dagger(y)^{m-n} \int_0^{x_\dagger(y)} s^{-m-1} H_y(s,y)\, ds + \int_{x_\dagger(y)}^\infty s^{-n-1} H_y(s,y)\, ds \right] \\
&> 0, \qquad (2.107)
\end{aligned}
$$

the inequality following from (2.100) and (2.24) in Assumption 3. Furthermore, the conclusion that $x_\dagger :]0, \infty[\to]0, x^\dagger(y)[$ is increasing implies that the definition $x_\dagger(0) = \lim_{y \downarrow 0} x_\dagger(y)$ exists.

To see (2.93), we use the upper bound in (2.28) in Assumption 3 to obtain

$$
q_\dagger(x,y) \geq x^{-n} \left[\frac{-(n-m)}{(\alpha - m)(n - \alpha)}\, \beta C y^{-(1-\beta)} x^\alpha - r\vartheta_2 \frac{n-m}{nm} \right]. \qquad (2.108)
$$

This inequality implies that $x_\dagger(y)$ is greater than the unique point at which the right hand side vanishes, namely,

$$x_\dagger(y) \geq \left[\frac{\sigma^2\vartheta_2}{\beta C}(\alpha - m)(n-\alpha)\right]^{\frac{1}{\alpha}} y^{\frac{1-\beta}{\alpha}}, \tag{2.109}$$

which establishes (2.93).

Now, fix any $(x,y) \in \mathcal{S}$ with $y > 0$, and consider the equation $F(x,y,z) = 0$ for $z > x$. Plainly,

$$F(x,y,x) = 0, \tag{2.110}$$

while a straightforward calculation involving the definition (2.63) of F and the definition of $R^{[H(\cdot)-rK]}(x,y)$ as in (2.64) yields

$$\frac{\partial}{\partial z}F(x,y,z) = -\frac{1}{\sigma^2}z^{m-n-1}q^\dagger(z,y). \tag{2.111}$$

However, these observations and (2.100) show that the equation $F(x,y,z) = 0$ has no solution $z > x$ if $x > x^\dagger(y)$. In view of this observation, (2.110), (2.111) and (2.100), we will prove part (a) of the lemma and the existence of a unique mapping $L_F : \mathcal{D} \to [0,\infty[$ such that $x < L_F(x,y)$ and $F(x,y,L_F(x,y)) = 0$ if we show that

$$\lim_{z\to\infty} F(x,y,z) \begin{cases} > 0, & \text{for } x \in]0, x_\dagger(y)[, \\ < 0, & \text{for } x \in]x_\dagger(y), x^\dagger(y)[. \end{cases} \tag{2.112}$$

To this end, we use the upper bound in (2.28) in Assumption 3 and the fact that $q^\dagger(x^\dagger(y),y) = 0$ to calculate

$$\begin{aligned}
&\lim_{z\to\infty} z^{m-n}\int_0^z s^{-m-1}[H(s,y) - rK]\,ds \\
&\quad = \lim_{z\to\infty} z^{m-n}\int_{x^\dagger(y)}^z s^{-m-1}[H(s,y) - rK]\,ds \\
&\quad \leq \lim_{z\to\infty} z^{m-n}\int_{x^\dagger(y)}^z s^{-m-1}[\beta C s^\alpha y^{-(1-\beta)} - r\vartheta_2]\,ds \\
&\quad = \lim_{z\to\infty}\left[\frac{\beta C y^{-(1-\beta)}}{\alpha - m}(z^{\alpha-n} - (x^\dagger(y))^{\alpha-m}z^{m-n}) + \frac{r\vartheta_2}{m}(z^{-n} - (x^\dagger(y))^{-m}z^{m-n})\right] \\
&\quad = 0,
\end{aligned} \tag{2.113}$$

the last equality following because $m < 0 < \alpha < n$. Similarly, we use the lower bound in (2.28) in Assumption 3 to obtain

$$
\begin{aligned}
\lim_{z\to\infty} & z^{m-n} \int_0^z s^{-m-1}[H(s,y) - rK]\, ds \\
&\geq \lim_{z\to\infty} -[C + rK] z^{m-n} \int_0^z s^{-m-1}\, ds \\
&= \lim_{z\to\infty} \frac{C + rK}{m} z^{-n} \\
&= 0. \qquad (2.114)
\end{aligned}
$$

These calculations and the definition of $R^{[H(\cdot)-rK]}$ in (2.64) imply that $\lim_{z\to\infty} z^{-n} R^{[H(\cdot)-rK]}(z,y) = 0$, which, combined with the definition of F and $q_\dagger$, implies

$$
\lim_{z\to\infty} F(x,y,z) = \frac{1}{\sigma^2(n-m)} q_\dagger(x,y). \qquad (2.115)
$$

However, this limit and (2.106) imply (2.112). Furthermore, a careful inspection of equations (2.100), (2.106), (2.111) and (2.115) reveals that (2.97) is true.

Finally, to show (2.96), we first differentiate $F(x,y,L_F(x,y)) = 0$ with respect to x to obtain

$$
\frac{\partial}{\partial x} L_F(x,y) = \frac{x^{m-n-1} q^\dagger(x,y)}{L_F^{m-n-1}(x,y) q^\dagger(L_F(x,y),y)} < 0, \quad \text{for all } (x,y) \in \operatorname{int} \mathcal{D} \qquad (2.116)
$$

the inequality following because $x < x^\dagger(y) < L_F(x,y)$ and $q^\dagger(x^\dagger(y),y) = 0$. Next, we observe that the definition of F in (2.63) implies

$$
F_y(x,y,z) = z^{-n} R^{[H_y]}(z,y) - x^{-n} R^{[H_y]}(x,y), \qquad (2.117)
$$

while the definition of $R^{[H_y]}$, which can be easily deduced from (2.64), implies

$$
\begin{aligned}
\frac{d}{dx} x^{-n} R^{[H_y]}(x,y) &= -\frac{1}{\sigma^2} x^{m-n-1} \int_0^x s^{-m-1} H_y(s,y)\, ds \\
&> 0, \qquad (2.118)
\end{aligned}
$$

the inequality following thanks to (2.24) in Assumption 3. However, these calculations and the fact that $x < x^\dagger(y) < L_F(x,y)$ show that

$$
F_y(x,y,L_F(x,y)) > 0, \quad \text{for all } (x,y) \in \operatorname{int} \mathcal{D}, \qquad (2.119)
$$

while (2.111) and (2.100) imply

$$F_z(x,y,L_F(x,y)) = -\frac{1}{\sigma^2} L_F^{m-n-1}(x,y) q^\dagger(L_F(x,y),y)$$
$$< 0, \quad \text{for all } (x,y) \in \operatorname{int}\mathcal{D}. \tag{2.120}$$

In view of these inequalities, we can differentiate $F(x,y,L_F(x,y)) = 0$ with respect to y to derive the second inequality in (2.96), and the proof is complete. □

The next result is concerned with a similar study of the function Φ defined by (2.69).

Lemma 18 *There exists a strictly increasing, C^1 function $\chi_\dagger : [0,\infty[\to \mathbb{R}_+$ such that given any $(x,y) \in \mathcal{S}$, the equation $\Phi(x,y,p) = 0$ has a solution $p > y$ if and only if $x > \chi_\dagger(y)$. In particular, there exists a unique mapping*

$$L_\Phi : \{(x,y) \in \mathcal{S} : \ x > \chi_\dagger(y)\} \to \mathcal{S} \tag{2.121}$$

such that

$$(x^\dagger)^{[-1]}(x) \le L_\Phi(x,y) \quad \text{and} \quad \Phi(x,y,L_\Phi(x,y)) = 0, \tag{2.122}$$

where the function $x^\dagger$ is as in Lemma (17). This mapping satisfies

$$\frac{\partial}{\partial x} L_\Phi(x,y) > 0 \quad \text{and} \quad \frac{\partial}{\partial y} L_\Phi(x,y) < 0, \quad \text{for all } y \ge 0 \text{ and } x > \chi_\dagger(y), \tag{2.123}$$

and there exists a function $\chi^\dagger : [0,\infty[\to \mathbb{R}_+$ such that $\chi_\dagger(y) < \chi^\dagger(y)$, for all $y \ge 0$, and, if we define

$$\mathcal{U} = \left\{(x,y) \in \mathcal{S} : \ x \in [\chi_\dagger(y), \chi^\dagger(y)]\right\},$$

then $L_\Phi(x,y) \in \mathcal{D}$, where $\mathcal{D}$ is defined by (2.94), if and only if $(x,y) \in \mathcal{U}$.

Proof. Fix any $(x,y) \in \mathcal{S}$, and define

$$\mathcal{D}_l = \{(x,y) \in \mathcal{S} : \ x \le x^\dagger(y)\}, \tag{2.124}$$

where $x^\dagger$ is as in Lemma 17. With regard to this definition, the fact that $m < 0$, (2.92) and (2.100), we calculate

$$\Phi(x, y, y) = \frac{rc}{m} < 0, \tag{2.125}$$

$$\frac{\partial}{\partial p}\Phi(x, y, p) = x^m q^\dagger(x, p) \begin{cases} < 0, & \text{for } x < x^\dagger(p), \\ = 0, & \text{for } x = x^\dagger(p), \\ > 0, & \text{for } x > x^\dagger(p), \end{cases} \tag{2.126}$$

and

$$\begin{aligned} \frac{\partial}{\partial x}\Phi(x, y, p) &= \int_y^p \left\{ m x^{m-1} \int_0^x s^{-m-1}[H(s, u) - rK]\, ds + x^{-1}[H(x, u) - rK] \right\} du \\ &= \int_y^p x^{m-1} \int_0^x s^{-m} H_x(s, u)\, ds\, du \\ &> 0, \end{aligned} \tag{2.127}$$

the inequality following thanks to (2.23) in Assumption 3. Now, (2.126) implies that the function $\Phi(x, y, \cdot)$ has a global maximum at y if $(x, y) \in \mathcal{D}_l$ and at $x^\dagger(p)$ if $(x, y) \in \mathcal{S} \backslash \mathcal{D}_l$. Combining this observation with (2.125), we can see that the equation $\Phi(x, y, p) = 0$ has a solution $p > y$ if and only if $(x, y) \in \mathcal{S} \backslash \mathcal{D}_l$ and $\Phi(x, y, (x^\dagger)^{[-1]}(x)) > 0$. However, this conclusion and (2.126) imply that there exists a strictly increasing function $\chi_\dagger : [0, \infty[\to \mathbb{R}$ and a mapping L_Φ satisfying (2.121)–(2.122) if and only if

$$\frac{\partial}{\partial x}\Phi(x, y, (x^\dagger)^{[-1]}(x)) > 0, \tag{2.128}$$

and

$$\lim_{x \to \infty} \Phi(x, y, (x^\dagger)^{[-1]}(x)) > 0. \tag{2.129}$$

Inequality (2.128) follows immediately once we observe that (2.126) and (2.127) imply

$$\frac{\partial}{\partial x}\Phi(x, y, (x^\dagger)^{[-1]}(x)) = \int_y^{(x^\dagger)^{[-1]}(x)} x^{m-1} \int_0^x s^{-m} H_x(s, u)\, ds\, du > 0. \tag{2.130}$$

To see (2.129), we note that (2.23) in Assumption 3 implies that, given any constant $N > 0$, there exists $x_1 > 0$ such that $H(x, y) - rK \geq N$, for all $x \geq x_1$. Given any such choice of constants,

$$
\begin{aligned}
&\lim_{x\to\infty} x^m \int_0^x s^{-m-1}[H(s,y) - rK]\,ds \\
&\quad \geq \lim_{x\to\infty} x^m \left[\int_0^{x_1} s^{-m-1}[H(s,y) - rK]\,ds - \frac{N}{m}x^{-m} + \frac{N}{m}x_1^{-m} \right] \\
&\quad = -\frac{N}{m},
\end{aligned}
\tag{2.131}
$$

the last equality following because $m < 0$. Since $N > 0$ is arbitrary, this calculation implies

$$
\lim_{x\to\infty} x^m \int_0^x s^{-m-1}[H(s,y) - rK]\,ds = \infty. \tag{2.132}
$$

However, in view of (2.132) and the fact that $\lim_{x\to\infty}(x^\dagger)^{[-1]}(x) = \infty$ (recall that the domain of $x^\dagger$ is the whole of $\mathbb{R}_+$), we can deduce that

$$
\begin{aligned}
\lim_{x\to\infty} \Phi(x,y,(x^\dagger)^{[-1]}(x)) &= \lim_{x\to\infty} \left[\int_y^{(x^\dagger)^{[-1]}(x)} x^m \int_0^x s^{-m-1}[H(s,u) - rK]\,ds\,du + \frac{rc}{m} \right] \\
&= \infty,
\end{aligned}
\tag{2.133}
$$

which establishes (2.129).

Now, to establish (2.123), we first differentiate $\Phi(x,y,L_\Phi(x,y)) = 0$ with respect to x to obtain, for all $y \geq 0$ and $x > \chi_\dagger(y)$,

$$
\frac{\partial}{\partial x} L_\Phi(x,y) = -\frac{\Phi_x(x,y,L_\Phi(x,y))}{\Phi_p(x,y,L_\Phi(x,y))} > 0, \tag{2.134}
$$

the inequality following thanks to (2.126) and (2.127) and the fact that $L_\Phi(x,y) \in \operatorname{int}\mathcal{D}_l$, which implies that $x < x^\dagger(L_\Phi(x,y))$. Also, differentiating $\Phi(x,y,L_\Phi(x,y)) = 0$ with respect to y yields

$$
\frac{\partial}{\partial y} L_\Phi(x,y) = \frac{q^\dagger(x,y)}{q^\dagger(x,L_\Phi(x,y))} < 0, \tag{2.135}
$$

the inequality following thanks to (2.100) and the inequalities $y < x_\dagger^{[-1]}(x) < L_\Phi(x,y)$.

With regard to the structure of the function Φ that we have studied above, the existence of the function $\chi^\dagger$ will follow if we prove that, given any $y \geq 0$,

$$
\lim_{x\to\infty} \Phi(x,y,x_\dagger^{[-1]}(x)) > 0. \tag{2.136}
$$

To this end, we consider any $x > 0$ and $p > y > y_1$, where y_1 is as in (2.30) in Assumption 3, and we calculate

$$\begin{aligned}\Phi(x,y,p) &\geq \int_y^p x^m \int_0^x s^{-m-1}[\beta\Lambda s^\alpha u^{-(1-\beta)} - rK]ds\, du + \frac{rc}{m} \\ &= \left[\frac{\Lambda\zeta}{\alpha-m}x^\alpha p^{-(1-\beta)} + \frac{rK}{m}\right]p + \left[\frac{\Lambda(1-\zeta)}{\alpha-m}p^\beta - \frac{\Lambda}{\alpha-m}y^\beta\right]x^\alpha - \frac{rK}{m}y + \frac{rc}{m},\end{aligned} \tag{2.137}$$

where ζ is a constant. Now, (2.93) and the identity $\sigma^2 mn = -r$ imply

$$\frac{\Lambda\zeta}{\alpha-m}x^\alpha[x_\dagger^{[-1]}(x)]^{-(1-\beta)} + \frac{rK}{m} \geq \left[\Lambda\zeta - \frac{K}{\vartheta_2}\frac{n\beta}{n-\alpha}C\right]\frac{\sigma^2\vartheta_2(n-\alpha)}{\beta C}, \tag{2.138}$$

while (2.29) in Assumption 3 implies that there exists $\zeta \in]0,1[$ such that the right hand side of this inequality is strictly positive. However, for such a choice of ζ, (2.137) and the fact that $\lim_{x\to\infty} x_\dagger^{[-1]}(x) = \infty$ imply (2.136), and the proof is complete. □

Proof of Lemma 15. With regard to the definitions and the properties of the sets $\mathcal{D}$, $\mathcal{U}$ and the mappings L_F, L_Φ in Lemmas 17 and 18, we define

$$\begin{aligned}\mathcal{L}_F A &= \{(L_F(x,y),y) :\ (x,y) \in A\}, \quad \text{for } A \subseteq \mathcal{D}, \\ \mathcal{L}_\Phi B &= \{(L_\Phi(x,y),y) :\ (x,y) \in B\}, \quad \text{for } B \subseteq \mathcal{U}, \\ x_1^{(0)} &= x_\dagger, \quad x_2^{(0)} = x^\dagger, \quad x_3^{(0)} = \chi_\dagger, \quad x_4^{(0)} = \chi^\dagger, \\ \mathcal{D}^{(0)} &= \mathcal{D} \equiv \left\{(x,y) \in \mathcal{S} :\ x \in [x_1^{(0)}(y), x_2^{(0)}(y)]\right\}\end{aligned}$$

and

$$\mathcal{U}^{(0)} = \mathcal{U} \equiv \left\{(x,y) \in \mathcal{S} :\ x \in [x_3^{(0)}(y), x_4^{(0)}(y)]\right\},$$

and we observe that

$$x_1^{(0)}(y) < x_2^{(0)}(y) < x_3^{(0)}(y) < x_4^{(0)}(y), \quad \text{for all } y \geq 0, \tag{2.139}$$

$$\lim_{y\to\infty} x_i^{(0)}(y) = \infty, \quad \text{for } i = 1,2,3,4, \tag{2.140}$$

$$\mathcal{L}_F\mathcal{D}^{(0)} \supset \mathcal{U}^{(0)} \text{ and } \mathcal{L}_\Phi\mathcal{U}^{(0)} = \left\{(x,y) \in \mathcal{D}^{(0)} :\ x \geq x_3^{(0)}(0)\right\}. \tag{2.141}$$

To proceed further, we appeal to an inductive argument, and we assume that we have found strictly increasing functions $x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, x_4^{(k)} : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$x_1^{(k)}(y) < x_2^{(k)}(y) < x_3^{(k)}(y) < x_4^{(k)}(y), \quad \text{for all } y \geq 0, \tag{2.142}$$

$$\lim_{y\to\infty} x_i^{(k)}(y) = \infty, \quad \text{for } i = 1, 2, 3, 4, \tag{2.143}$$

and, if $\mathcal{D}^{(k)}$, $\mathcal{U}^{(k)}$ are the sets defined by

$$\mathcal{D}^{(k)} = \left\{(x,y) \in \mathcal{S} : \ x \in [x_1^{(k)}(y), x_2^{(k)}(y)]\right\},$$
$$\mathcal{U}^{(k)} = \left\{(x,y) \in \mathcal{S} : \ x \in [x_3^{(k)}(y), x_4^{(k)}(y)]\right\},$$

then

$$\mathcal{L}_F \mathcal{D}^{(k)} \supset \mathcal{U}^{(k)} \quad \text{and} \quad \mathcal{L}_\Phi \mathcal{U}^{(k)} = \left\{(x,y) \in \mathcal{D}^{(k)} : \ x \geq x_3^{(k)}(0)\right\}. \tag{2.144}$$

With regard to the properties of the function $L_F(\cdot, y)$ established in Lemma 17, there exist functions $x_1^{(k+1)}, x_2^{(k+1)} : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$x_1^{(k)}(y) < x_1^{(k+1)}(y) < x_2^{(k+1)}(y) < x_2^{(k)}(y), \quad \text{for all } y \geq 0, \tag{2.145}$$

$$\lim_{y\to\infty} x_1^{(k+1)}(y) = \lim_{y\to\infty} x_2^{(k+1)}(y) = \infty, \tag{2.146}$$

and, if we define

$$\mathcal{D}^{(k+1)} = \left\{(x,y) \in \mathcal{S} : \ x \in [x_1^{(k+1)}(y), x_2^{(k+1)}(y)]\right\}, \tag{2.147}$$

then

$$\mathcal{L}_F \mathcal{D}^{(k+1)} = \mathcal{U}^{(k)}. \tag{2.148}$$

Similarly, the properties of the function L_Φ in Lemma 18 imply that there exist functions $x_2^{(k+1)}, x_3^{(k+1)} : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$x_3^{(k)}(y) < x_3^{(k+1)}(y) < x_4^{(k+1)}(y) < x_4^{(k)}(y), \quad \text{for all } y \geq 0, \tag{2.149}$$

$$\lim_{y\to\infty} x_3^{(k+1)}(y) = \lim_{y\to\infty} x_4^{(k+1)}(y) = \infty, \tag{2.150}$$

$$\mathcal{L}_\Phi \mathcal{U}^{(k+1)} = \left\{(x,y) \in \mathcal{D}^{(k+1)} : \ x \geq x_3^{(k+1)}(0)\right\}, \tag{2.151}$$

where $\mathcal{U}^{(k+1)}$ is given by

$$\mathcal{U}^{(k+1)} = \left\{ (x,y) \in \mathcal{S} : \ x \in [x_3^{(k+1)}(y), x_4^{(k+1)}(y)] \right\}. \tag{2.152}$$

By construction, the functions $x_i^{(k+1)}$, $i = 1,2,3,4$, and the sets $\mathcal{D}^{(k+1)}$, $\mathcal{U}^{(k+1)}$ have all of the properties assumed for the functions $x_i^{(k)}$, $i = 1,2,3,4$, and the domains $\mathcal{D}^{(k)}$, $\mathcal{U}^{(k)}$ (see (2.142)–(2.144)), and which are shared by the corresponding entities when $k = 0$ (see (2.139–(2.141)). By induction, it follows that there exist sequences of functions $(x_1^{(k)})$, $(x_2^{(k)})$, $(x_3^{(k)})$, $(x_4^{(k)})$ and subsets $(\mathcal{D}^{(k)})$, $(\mathcal{U}^{(k)})$ of $\mathcal{S}$ satisfying (2.142)–(2.144). Since $(x_1^{(k)})$, $(x_3^{(k)})$ (resp., $(x_2^{(k)})$, $(x_4^{(k)})$) are strictly increasing (resp., decreasing) sequences of functions,

$$\hat{x}_i(y) = \lim_{k\to\infty} x_i^{(k)}(y), \quad \text{for } y \geq 0 \text{ and } i = 1,2,3,4, \tag{2.153}$$

define increasing functions $\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4 : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\hat{x}_1$, $\hat{x}_3$ are lower semi-continuous, $\hat{x}_2$, $\hat{x}_4$ are upper semi-continuous,

$$\begin{gathered} \hat{x}_1(y) \leq \hat{x}_2(y) < \hat{x}_3(y) \leq \hat{x}_4(y), \quad \text{for all } y \geq 0, \\ \lim_{y\to\infty} \hat{x}_i(y) = \infty, \quad \text{for } i = 1,2,3,4, \end{gathered} \tag{2.154}$$

and the sets

$$\begin{aligned} \mathbb{G}_1 &= \bigcap_{k=0}^{\infty} \mathcal{D}^{(k)} \equiv \{(x,y) \in \mathcal{S} : \ \hat{x}_1(y) \leq x \leq \hat{x}_2(y)\}, \\ \mathbb{G}_0 &= \bigcap_{k=0}^{\infty} \mathcal{U}^{(k)} \equiv \{(x,y) \in \mathcal{S} : \ \hat{x}_3(y) \leq x \leq \hat{x}_4(y)\} \end{aligned} \tag{2.155}$$

are non-empty and closed. Moreover, (2.148) and (2.151) imply

$$\mathcal{L}_F \mathbb{G}_1 = \mathbb{G}_0 \quad \text{and} \quad \mathcal{L}_{\Phi} \mathbb{G}_0 = \left\{ (x,y) \in \mathcal{S} : \ x \geq \lim_{k\to\infty} x_3^{(k)}(0) \right\},$$

respectively, while the fact that $L_F(\cdot, y)$ and $L_\Phi(x, \cdot)$ are both decreasing implies

$$\begin{gathered} \mathcal{L}_F \operatorname{graph}(\hat{x}_1) = \operatorname{graph}(\hat{x}_4), \qquad \mathcal{L}_F \operatorname{graph}(\hat{x}_2) = \operatorname{graph}(\hat{x}_3), \\ \mathcal{L}_\Phi \operatorname{graph}(\hat{x}_3) \subset \operatorname{graph}(\hat{x}_2) \quad \text{and} \quad \mathcal{L}_\Phi \operatorname{graph}(\hat{x}_4) \subset \operatorname{graph}(\hat{x}_1). \end{gathered}$$

These inclusions imply that, if we define

$$x^{\circ} = \hat{x}_3(0), \quad G_1(x) = \hat{x}_2^{[-1]}(x) \quad \text{and} \quad G_0(x) = \hat{x}_3^{[-1]}(x), \quad \text{for } x \geq x^{\circ},$$

then G_1 and G_0 satisfy (2.62), for all $y \geq 0$, and (2.68), for all $x \geq x^{\circ}$. Moreover, since the functions F and Φ are C^1, the functions G_1, G_0 are C^1. At this point, we should note that the choice of G_1, G_0 made above plainly appears to be non-unique, which is due to the fact that we have not managed to prove that the sets $\mathbb{G}_0$ and $\mathbb{G}_1$ have empty interior.

Now, consider (2.70), and note that the upper bound in (2.28) in Assumption 3 and the inequalities $\alpha < \frac{\alpha}{1-\beta} < n$ imply

$$0 < A(y) \leq \frac{\beta C}{\sigma^2(\alpha - m)(n-\alpha)} \int_y^{\infty} u^{-(1-\beta)} \left(G_1^{[-1]}(u)\right)^{-(n-\alpha)} du \tag{2.156}$$

Now, fix any $\varepsilon_0 > 0$ such that

$$\varepsilon_0 < n - \frac{\alpha}{1-\beta} < n - \alpha.$$

Using the fact that $G_1^{[-1]}$ is increasing and the estimate provided by (2.71), we calculate

$$\begin{aligned}\int_y^{\infty} u^{-(1-\beta)} \left(G_1^{[-1]}(u)\right)^{-(n-\alpha)} du &\leq \left(G_1^{[-1]}(y)\right)^{-\varepsilon_0} \int_y^{\infty} u^{-(1-\beta)} \left(G_1^{[-1]}(u)\right)^{-(n-\alpha-\varepsilon_0)} du \\ &\leq \frac{\alpha C_{71}}{(1-\beta)(n-\varepsilon_0)-\alpha} \left(G_1^{[-1]}(y)\right)^{-\varepsilon_0} y^{-[(1-\beta)(n-\varepsilon_0)-\alpha]/\alpha},\end{aligned}$$

where $C_{71} > 0$ is a constant, which implies

$$\int_y^{\infty} u^{-(1-\beta)} \left(G_1^{[-1]}(u)\right)^{-(n-\alpha)} du \leq \frac{\alpha C_{71}}{(1-\beta)(n-\varepsilon_0)-\alpha} \left(G_1^{[-1]}(y)\right)^{-\varepsilon_0}, \quad \text{for all } y \geq 1. \tag{2.157}$$

Also, the fact that $G_1^{[-1]}$ is increasing implies that, given any $y < 1$,

$$\begin{aligned}\left(G_1^{[-1]}(y)\right)^{n} \int_y^{1} u^{-(1-\beta)} \left(G_1^{[-1]}(u)\right)^{-(n-\alpha)} du &\leq \left(G_1^{[-1]}(y)\right)^{\alpha} \int_y^{1} u^{-(1-\beta)} du \\ &\leq \frac{1}{\beta} \left(G_1^{[-1]}(1)\right)^{\alpha}.\end{aligned} \tag{2.158}$$

However, (2.156)–(2.158) imply

$$\begin{aligned}
A(y)\left(G_1^{[-1]}(y)\right)^n \leq & \frac{\beta C}{\sigma^2(n-m)(n-\alpha)}\left[\frac{\alpha C_{71}}{(1-\beta)(n-\varepsilon_0)-\alpha}\left(\left(G_1^{[-1]}(y)\right)^{n-\varepsilon_0} 1_{\{y\geq 1\}}\right)\right. \\
& \left. + \left(G_1^{[-1]}(1)\right)^{n-\varepsilon_0} 1_{\{y<1\}} + \frac{1}{\beta}\left(G_1^{[-1]}(1)\right)^{\alpha} 1_{\{y<1\}}\right] \\
= & C_{72}\left(1+\left(G_1^{[-1]}(y)\right)^{n-\varepsilon_0}\right), \quad \text{for all } y \geq 0, \qquad (2.159)
\end{aligned}$$

where $C_{72} > 0$ is a constant.

To proceed further, fix any $(x, y) \in \mathcal{W}$ such that $x \leq G_1^{[-1]}(0)$ or $x > G_1^{[-1]}(0)$ and $y > G_1(x)$. For such a point,

$$\begin{aligned}
w(x,y) &\leq w(G_1^{[-1]}(y), y) \\
&= A(y)\left[G_1^{[-1]}(y)\right]^n + R^{[h]}(G_1^{[-1]}(y), y) \\
&\leq C_{73}\left(1 + y + \left[G_1^{[-1]}(y)\right]^{n-\varepsilon_0\wedge\vartheta_1} + \left[G_1^{[-1]}(y)\right]^{\alpha} y^{\beta}\right), \qquad (2.160)
\end{aligned}$$

the first inequality following because $w(\cdot, y)$ is increasing, and the second one following thanks to (2.159) and the upper bound in Lemma 12.

For $(x, y) \in \mathcal{W}$ such that $y < G_1(x)$, the fact that w satisfies the HJB equation (2.49) implies

$$w(x, G_0(x)) \geq w(x,y) - K[y - G_0(x)] - c, \qquad (2.161)$$

which, combined with the identity

$$w(x, G_0(x)) = w(x, G_1(x)) - K[G_1(x) - G_0(x)] - c \qquad (2.162)$$

that is true by construction, implies

$$\begin{aligned}
w(x,y) &\leq w(x, G_1(x)) + Ky \\
&= A(G_1(x))x^n + R(x, G_1(x)) + Ky \\
&\leq C_{74}\left(1 + x^{n-\varepsilon_0\wedge\vartheta_1} + G_1(x) + x^{\alpha}[G_1(x)]^{\beta}\right) \\
&\leq C_{74}\left(1 + x^{n-\varepsilon_0\wedge\vartheta_1} + x^{\frac{\alpha}{1-\beta}}\right), \qquad (2.163)
\end{aligned}$$

where $C_{74} > 0$ is a constant. The second inequality here follows thanks to (2.159) and Lemma 12, while the third one is true because of the estimate for G_1 provided by

(2.71). Also, if $(x,y) \in \mathcal{I}$, then the expression for w given by (2.72) implies that $w(x,y)$ satisfies (2.163) as well. However, (2.160) and (2.163) establish the upper estimate in (2.73).

To show that w satisfies the lower bound in (2.73), we first observe that the positivity of A and the lower bound in Lemma 12 imply that

$$w(x,y) \geq -C_1(1+y), \quad \text{for all } (x,y) \in \overline{\mathcal{W}}. \tag{2.164}$$

This estimate and the definition of w in $\mathcal{I}$, provided by (2.72), imply

$$\begin{aligned} w(x,y) &\geq -(C_1+K)\, G_1(x) - C_1 \\ &\geq -C_{75}\,(1 + x^{\alpha/(1-\beta)}), \quad \text{for all } (x,y) \in \mathcal{I}, \end{aligned} \tag{2.165}$$

the second inequality following thanks to (2.71). However, (2.164)–(2.165) establish the lower bound in (2.73).

To see that w is $C^{1,1}$ along the boundary G_1, we use the second identity in (2.72) along with (2.59) to calculate

$$\begin{aligned} w_x(x,y) &= w_x(x, G_1(x)) + [w_y(x, G_1(x)) - K]\frac{dG_1(x)}{dx} \\ &= w_x(x, G_1(x)). \end{aligned} \tag{2.166}$$

By construction, we will prove that w satisfies the HJB equation (2.49) if we show that

$$\sigma^2 x^2 w_{xx}(x,y) + bxw_x(x,y) - rw(x,y) + h(x,y) \leq 0, \quad \text{for all } (x,y) \in \mathcal{I}, \tag{2.167}$$

$$-w(x,y) - c + \sup_{z>0}[w(x,y+z) - Kz] \leq 0, \quad \text{for all } (x,y) \in \mathcal{W} \tag{2.168}$$

To this end, we fix any $(x,y) \in \mathcal{W}$ and we observe that (2.60) and the definition of F in (2.63) imply

$$\begin{aligned} w_y(x,y) - K &= A'(y)x^n + R^{[H(\cdot)-rK]}(x,y) \\ &= \left[-(G_0^{[-1]}(y))^{-n} R^{[H(\cdot)-rK]}(G_0^{[-1]}(y), y) + x^{-n} R^{[H(\cdot)-rK]}(x,y)\right] x^n \\ &= -F(x, y, G_0^{[-1]}(y))x^n \\ &\begin{cases} < 0, & \text{for } x < G_1^{[-1]}(y), \\ > 0, & \text{for } x \in]G_1^{[-1]}(y), G_0^{[-1]}(y)[. \end{cases} \end{aligned} \tag{2.169}$$

To see how the inequalities follow, we note that

$$\frac{\partial F(x,y,z)}{\partial x} = \frac{1}{\sigma^2} x^{m-n-1} q^\dagger(x,y), \tag{2.170}$$

and that $q(x^\dagger(y), y) = 0$. Then, since $G_1^{[-1]}(y) < x^\dagger(y) < G_0^{[-1]}(y)$, we must have

$$\frac{\partial}{\partial x} F(x, y, G_0^{[-1]}(y)) \begin{cases} < 0, & \text{for } x < G_1^{[-1]}(y) \text{ and } x \in \left]G_1^{[-1]}(y), x^\dagger(y)\right[, \\ > 0, & \text{for } x \in \left]x^\dagger(y), G_0^{[-1]}(y)\right[. \end{cases} \tag{2.171}$$

Combining this with $F(G_0^{[-1]}(y), y, G_0^{[-1]}(y)) = F(G_1^{[-1]}(y), y, G_0^{[-1]}(y)) = 0$, we deduce that

$$F(x, y, G_0^{[-1]}(y)) \begin{cases} > 0, & \text{for } x < G_1^{[-1]}(y), \\ < 0, & \text{for } x \in \left]G_1^{[-1]}(y), G_0^{[-1]}(y)\right[. \end{cases} \tag{2.172}$$

Using (2.169), it is a tedious but straightforward exercise to show that (2.168) is satisfied.

To establish (2.167), and in view of the $C^{2,2}$ continuity of w in the interior of $\mathcal{W}$, we can differentiate $w_y(x, G_1(x)) = K$ with respect to x to obtain

$$w_{xy}(x, G_1(x)) = -w_{yy}(x, G_1(x))\, G_1'(x) \le 0, \tag{2.173}$$

the inequality following because $w_{yy}(x, G_1(x)) \ge 0$, which is true thanks to (2.169), and the fact that G_1 is strictly increasing. Now, with regard to (2.166), we can see that

$$\begin{aligned} w_{xx}(x,y) &= w_{xx}(x, G_1(x)) + w_{xy}(x, G_1(x))\, G_1'(x) \\ &\le w_{xx}(x, G_1(x)), \quad \text{for all } (x,y) \in \mathcal{I}. \end{aligned} \tag{2.174}$$

combining this inequality with (2.166) and (2.72), we can see that (2.167) is implied by

$$\begin{aligned} \sigma^2 x^2 w_{xx}(x, G_1(x)) &+ bx w_x(x, G_1(x)) - rw(x, G_1(x)) \\ &+ rK[G_1(x) - y] + h(x,y) + rc \le 0, \quad \text{for } (x,y) \in \mathcal{I}, \end{aligned} \tag{2.175}$$

which is equivalent to

$$-\int_{y}^{G_1(x)} [H(x,u) - rK]\, du + rc \leq 0. \tag{2.176}$$

However, this is true because, by (2.68), we have

$$\begin{aligned} rc &= -m \int_{G_0(x)}^{G_1(x)} x^m \int_0^x s^{-m-1}[H(s,u) - rK]\, ds\, du \\ &\leq -m \int_{G_0(x)}^{G_1(x)} x^m \int_0^x s^{-m-1}[H(x,u) - rK]\, ds\, du \\ &= \int_{G_0(x)}^{G_1(x)} [H(x,u) - rK]\, du, \end{aligned} \tag{2.177}$$

the inequality here following from (2.23). Substituting this into (2.176) yields

$$\begin{aligned} -\int_{y}^{G_0(x)} [H(x,u) - rK]\, du &\leq -\int_{y}^{G_0(x)} [H(x, G_0(x)) - rK]\, du \\ &\leq 0, \end{aligned} \tag{2.178}$$

the first inequality due to (2.24) and the second to the fact that $H(x, G_0(x)) - rK \geq H(x, (x^{\dagger})^{[-1]}) - rK = 0$, and the proof is complete. $\square$

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